

**MODELING AND ANALYSIS OF STOCHASTIC
INVENTORY SYSTEM IN MULTI-ECHELON WITH
RETRIAL DEMANDS**

**Thesis submitted to Madurai Kamaraj for the degree of
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IN
MATHEMATICS**

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CERTIFICATE

This is to certify that the thesis entitled “**Modeling And Analysis of Stochastic Inventory System In Multi-Echelon With Retrial Demands**” submitted by **Mr. R.SATHEESH KUMAR** (Registration No.:P3252), P.G. Assistant in Mathematics, Shri Renuga Vidhyalaya Matric. Hr. Sec. School, Lakshmipuram, Periyakulam, Theni District. for the award of the degree of Doctor of Philosophy in Mathematics is a bonafide record of research work carried out by him under my supervision.

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DECLARATION

I do hereby declare that this thesis entitled “**Modeling And Analysis of Stochastic Inventory System In Multi-Echelon With Retrial Demands**” has been originally carried out by me under the guidance and supervision of Dr. K. KRISHNAN, Assistant Professor, PG and Research Department of Mathematics, Cardamom Planters’ Association College, Bodinayakanur – 625 513, Tamil Nadu, India and that this work has not been submitted elsewhere for any other degree or diploma.

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(R.SATHEESH KUMAR)

Dedicated

to

My Family & Friends

ABSTRACT

The title of the Thesis is **Modeling and Analysis of Stochastic Inventory System in Multi-Echelon with Retrial Demands**. The proposed thesis consists of seven chapters.

Chapter-1 presents, some preliminary concepts in Inventory System and its developments, importance, different type of decision-making, and Multi-echelon inventory system. Besides, a brief review of the literature relevant to the topic. An outline of the work done in the present thesis are also discussed.

In Chapter-2, some basic definitions and important concepts of Stochastic processes and their applications are given.

Chapter-3, deals with an integrated inventory control system that consists of one manufacturer (MF), one Distribution centre (DC), and one retailer. An (s,S) type inventory system with Poisson demand and exponentially distributed lead times are assumed at retailer and DC respectively. The demands occurring during the stock out period and the unsatisfied customer demand are permitted to enter into an orbit of size N. Such unsatisfied customer demand is known as Retrial demand. The above network model is studied for both Non-perishable and Perishable products. The steady state probability distribution of system states and the measures of system performance are obtained for both the models. Numerical examples with sensitivity analysis are provided to illustrate the proposed models.

Chapter-4 deals a continuous review inventory control system consists of a Manufacturer (MF), single Warehouse (WH), one Distribution Centre (DC) and n identical retailers system handling a single perishable product. A (s, Q) type inventory system with Poisson demand and exponentially distributed lead times for items are assumed at DC (middle echelon). An one-for-one type inventory policy is assumed at retailers node (lower echelon). Demands occurring during the stock out periods are assumed to be lost. The WH (upper echelon) replenishes their stocks from Manufacturer, which has abundant stocks for supply. The items are supplied to the Manufacturer in packs of Q items from the warehouse. The perishable product has the exponential decay of constant rate. Here the exposure of items occurs only at retailer node. The steady state

probability distribution and the operating characteristics are obtained explicitly. The required algorithm is designed and it is executed.

Chapter-5, deals with a simple supply chain that is modeled as a single warehouse and multiple retailer system handling a single product. In order to avoid the complexity, at the same time without loss of generality, we assumed identical demand pattern at each node. This restricts our study to design and analyze a tandem network of inventory, which is a building block for the whole supply chain system. A (s, S) type inventory system with Poisson demand and exponentially distributed lead times is assumed at retailer node. The items are supplied to the retailers and the direct customers (Direct demand), in packs of $Q (= S-s)$ items from the distribution center which has instantaneous replenishment facility from an abundant source (manufacturer). The unsatisfied customers at retailer are entered into the orbit of finite capacity N . These orbiting demands retry for their demand after a random time which is assumed to be exponential distribution. The transient and steady state probability distribution and the operating characteristics are obtained explicitly. The measures of system performance in the steady state are obtained. The required algorithm is designed and it is executed.

Chapter-6, deals with a continuous review inventory system in supply chain (consist of retailer, distributor and manufacturer) with Markovian demand. The operating policy is (s, S) policy, that is, the maximum inventory level is S and whenever the inventory level drops to s an order for $Q(S-s)$ units is placed. The items are supplied to the retailers and the direct customers (Direct demand), in packs of Q items from the distribution center which has instantaneous replenishment facility from an abundant source (manufacturer). The ordered items are received after a random time which is assumed to be exponential distribution. The demands that occur during stock out period are enter into the orbit of finite size N . These orbiting demands retry for their demand after a random time, which is assumed to be exponential distribution. The joint probability distribution of the inventory levels of both retailer and distributor and the number of customers in the orbit is obtained in the steady state case. Various system performance measures in the steady state are derived and the long-run total expected cost rate is calculated. Several instances of a numerical example, which provide insight into the behavior of the system, are presented.

In the last **Chapter-7**, the summary of the study, conclusions and scope for further work are given.

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CHAPTER 1

INTRODUCTION

1.1 Inventory System and Its Motivation

“Sorry, we’re out of that item.” How often have you heard that during shopping trips? In many of these cases, what you have encountered are stores that aren’t doing a very good job of managing their *inventories* (stocks of goods being held for future use or sale). They aren’t placing orders to replenish inventories soon enough to avoid shortages. These stores could benefit from the kinds of techniques of scientific inventory management that are described in this chapter.

It isn’t just retail stores that must manage inventories. In fact, inventories pervade the business world. Maintaining inventories is necessary for any company dealing with physical products, including manufacturers, wholesalers, and retailers. For example, manufacturers need inventories of the materials required to make their products. They also need inventories of the finished products awaiting shipment. Similarly, both wholesalers and retailers need to maintain inventories of goods to be available for purchase by customers.

$$\text{The total value of all inventory} \begin{cases} 0 & \text{if } z = 0 \\ K + cz & \text{if } z > 0 \end{cases}$$

including finished goods, partially finished goods, and raw materials in the United States is more than a *trillion* dollars. This is more than \$4,000 each for every man, woman, and child in the country. The costs associated with storing (“carrying”) inventory are also very large, perhaps a quarter of the value of the inventory. Therefore, the costs being incurred for the storage of inventory in the United States run into the hundreds of billions of dollars annually. Reducing storage costs by avoiding unnecessarily large inventories can enhance any firm’s competitiveness.

Some Japanese companies were pioneers in introducing the *just-in-time inventory system*—a system that emphasizes planning and scheduling so that the needed materials arrive “just-in-time” for their use. Huge savings are thereby achieved by reducing inventory levels to a bare minimum.

Many companies in other parts of the world also have been revamping the way in which they manage their inventories. The application of operations research techniques in this area (sometimes called *scientific inventory management*) is providing a powerful tool for gaining a competitive edge.

How do companies use operations research to improve their **inventory policy** for when and how much to replenish their inventory? They use **scientific inventory management** comprising the following steps:

1. Formulate a *mathematical model* describing the behavior of the inventory system.
2. Seek an *optimal* inventory policy with respect to this model.
3. Use a computerized *information processing system* to maintain a record of the current inventory levels.
4. Using this record of current inventory levels, apply the optimal inventory policy to signal when and how much to replenish inventory.

The mathematical inventory models used with this approach can be divided into two broad categories deterministic models and stochastic model according to the *predictability of demand* involved. The **demand** for a product in inventory is the number of units that will need to be withdrawn from inventory for some use (e.g., sales) during a specific period. If the demand in future periods can be forecast with considerable precision, it is reasonable to use an inventory policy that assumes that all forecasts will always be completely accurate. This is the case of *known demand* where a *deterministic* inventory model would be used. However, when demand cannot be predicted very well, it becomes necessary to use a *stochastic* inventory model where the demand in any period is a random variable rather than a known constant.

There are several basic considerations involved in determining an inventory policy that must be reflected in the mathematical inventory model. The following three sections present stochastic models, first under continuous review, then for a single period, and finally for a series of periods. The chapter concludes with a discussion of how scientific inventory management is being used in practice to deal with very large inventory systems, as illustrated by case studies at IBM and Hewlett-Packard.

1.2 Components of Inventory Models

Because inventory policies affect profitability, the choice among policies depends upon their relative profitability. Some of the costs that determine this profitability are (1) the ordering costs, (2) holding costs, and (3) shortage costs. Other relevant factors include (4) revenues, (5) salvage costs, and (6) discount rates. These six factors are described in turn below.

The cost of ordering an amount z (either through purchasing or producing this amount) can be represented by a function $c(z)$. The simplest form of this function is one that is directly proportional to the amount ordered, that is, $c(z)$ where c represents the unit price paid. Another common assumption is that $c(z)$ is composed of two parts: a term that is directly proportional to the amount ordered and a term that is a constant K for z positive and is 0 for $z=0$. For this case,

$$c(z) = \text{cost of ordering } z \text{ units} = \begin{cases} 0 & \text{if } z = 0 \\ K + cz & \text{if } z > 0 \end{cases}$$

The constant K includes the administrative cost of ordering or, when producing, the costs involved in setting up to start a production run.

The **holding cost** (sometimes called the *storage cost*) represents all the costs associated with the storage of the inventory until it is sold or used. Included are the cost of capital tied up, space, insurance, protection, and taxes attributed to storage. The holding cost can be assessed either continuously or on a period-by-period basis. In the latter case, the cost may be a function of the maximum quantity held during a period, the average amount held, or the quantity in inventory at the end of the period.

The **shortage cost** (sometimes called the *unsatisfied demand cost*) is incurred when the amount of the commodity required (demand) exceeds the available stock. This cost depends upon which of the following two cases applies.

In one case, called **backlogging**, the excess demand is not lost, but instead is held until it can be satisfied when the next normal delivery replenishes the inventory. For a firm incurring a temporary shortage in supplying its customers, the shortage cost then can be interpreted as the loss of customers goodwill and the subsequent

reluctance to do business with the firm, the cost of delayed revenue, and the extra administrative costs. For a manufacturer incurring a temporary shortage in materials needed for production (such as a shortage of speakers for assembly into television sets), the shortage cost becomes the cost associated with delaying the completion of the production process.

In the second case, called **no backlogging**, if any excess of demand over available stock occurs, the firm cannot wait for the next normal delivery to meet the excess demand. Either (1) the excess demand is met by a priority shipment, or (2) it is not met at all because the orders are canceled. For situation 1, the shortage cost can be viewed as the cost of the priority shipment. For situation 2, the shortage cost is the loss of current revenue from not meeting the demand plus the cost of losing future business because of lost goodwill.

Revenue may or may not be included in the model. If both the price and the demand for the product are established by the market and so are outside the control of the company, the revenue from sales (assuming demand is met) is independent of the firm's inventory policy and may be neglected. However, if revenue is neglected in the model, the *loss in revenue* must then be included in the shortage cost whenever the firm cannot meet the demand and the sale is lost. Furthermore, even in the case where demand is backlogged, the cost of the delay in revenue must also be included in the shortage cost. With these interpretations, revenue will not be considered explicitly in the remainder of this chapter.

The **salvage value** of an item is the value of a leftover item when no further inventory is desired. The salvage value represents the disposal value of the item to the firm, perhaps through a discounted sale. The negative of the salvage value is called the **salvage cost**. If there is a cost associated with the disposal of an item, the salvage cost may be positive. We assume hereafter that any salvage cost is incorporated into the *holding cost*.

Finally, the **discount rate** takes into account the time value of money. When a firm ties up capital in inventory, the firm is prevented from using this money for alternative purposes. For example, it could invest this money in secure investments, say, government bonds, and have a return on investment 1 year hence of, say, 7

percent. Thus, \$1 invested today would be worth \$1.07 in year 1, or alternatively, a \$1 profit 1 year hence is equivalent to $a = \$1/\1.07 today. The quantity a is known as the **discount factor**. Thus, in adding up the total profit from an inventory policy, the profit or costs 1 year hence should be multiplied by a ; in 2 years hence by a^2 ; and so on. (Units of time other than 1 year also can be used.) The total profit calculated in this way normally is referred to as the *net present value*.

In problems having short time horizons, a may be assumed to be 1 (and thereby neglected) because the current value of \$1 delivered during this short time horizon does not change very much. However, in problems having long time horizons, the discount factor must be included.

In using quantitative techniques to seek optimal inventory policies, we use the criterion of minimizing the total (expected) discounted cost. Under the assumptions that the price and demand for the product are not under the control of the company and that the lost or delayed revenue is included in the shortage penalty cost, minimizing cost is equivalent to maximizing net income. Another useful criterion is to keep the inventory policy simple, i.e., keep the rule for indicating *when to order* and *how much to order* both understandable and easy to implement. Most of the policies considered in this chapter possess this property.

As mentioned at the beginning of the chapter, inventory models are usually classified as either *deterministic* or *stochastic* according to whether the demand for a period is known or is a random variable having a known probability distribution. Another component of an inventory model is the **lead time**, which is the amount of time between the placement of an order to replenish inventory (through either purchasing or producing) and the receipt of the goods into inventory. If the lead time always is the same (a *fixed* lead time), then the replenishment can be scheduled just when desired.

Another classification refers to whether the current inventory level is being monitored continuously or periodically. In **continuous review**, an order is placed as soon as the stock level falls down to the prescribed reorder point. In **periodic review**, the inventory level is checked at discrete intervals, e.g., at the end of each week, and ordering decisions are made only at these times even if the inventory level dips below

the reorder point between the preceding and current review times. (In practice, a periodic review policy can be used to approximate a continuous review policy by making the time interval sufficiently small.)

1.3 Deterministic Continuous-Review Models

The most common inventory situation faced by manufacturers, retailers, and wholesalers is that stock levels are depleted over time and then are replenished by the arrival of a batch of new units. A simple model representing this situation is the following **Economic Order Quantity model** or, for short, the **EOQ model**. (It sometimes is also referred to as the *economic lot-size model*.)

Units of the product under consideration are assumed to be withdrawn from inventory continuously at a *known constant rate*, denoted by a , that is, the demand is a units per unit time. It is further assumed that inventory is replenished when needed by ordering (through either purchasing or producing) a batch of fixed size (Q units), where all Q units arrive simultaneously at the desired time. For the *basic EOQ model* to be presented first, the only costs to be considered are

K = setup cost for ordering one batch,

c = unit cost for producing or purchasing each unit,

h = holding cost per unit per unit of time held in inventory.

The objective is to determine when and by how much to replenish inventory so as to minimize the sum of these costs per unit time.

We assume *continuous review*, so that inventory can be replenished whenever the inventory level drops sufficiently low. We shall first assume that shortages are not allowed (but later we will relax this assumption). With the fixed demand rate, shortages can be avoided by replenishing inventory each time the inventory level drops to zero, and this also will minimize the holding cost.

1.4 A Deterministic Periodic-Review Model

The preceding section explored the basic EOQ model and some of its variations. The results were dependent upon the assumption of a constant demand

rate. When this assumption is relaxed, i.e., when the amounts that need to be withdrawn from inventory are allowed to vary from period to period, the *EOQ formula* no longer ensures a minimum-cost solution.

Consider the following periodic-review model. Planning is to be done for the next n periods regarding how much (if any) to produce or order to replenish inventory at the beginning of each of the periods. (The order to replenish inventory can involve either *purchasing* the units or *producing* them, but the latter case is far more common with applications of this model, so we mainly will use the terminology of *producing* the units.) The demands for the respective periods are *known* (but *not* the same in every period) and are denoted by

$$r_i = \text{demand in period } i, \quad \text{for } i = 1, 2, \dots, n.$$

These demands must be met on time. There is no stock on hand initially, but there is still time for a delivery at the beginning of period 1.

The costs included in this model are similar to those for the basic EOQ model:

K = setup cost for producing or purchasing any units to replenish inventory at beginning of period,

c = unit cost for producing or purchasing each unit,

h = holding cost for each unit left in inventory at end of period.

Note that this holding cost h is assessed only on inventory left at the end of a period. There also are holding costs for units that are in inventory for a portion of the period before being withdrawn to satisfy demand. However, these are *fixed* costs that are independent of the inventory policy and so are not relevant to the analysis. Only the *variable* costs that are affected by which inventory policy is chosen, such as the extra holding costs that are incurred by carrying inventory over from one period to the next, are relevant for selecting the inventory policy.

By the same reasoning, the unit cost c is an irrelevant fixed cost because, over all the time periods, all inventory policies produce the same number of units at the same cost. Therefore, c will be dropped from the analysis hereafter.

The objective is to minimize the total cost over the n periods. This is accomplished by ignoring the fixed costs and minimizing the total variable cost over the n periods.

1.5 A Stochastic Continuous-Review Model

We now turn to *stochastic* inventory models, which are designed for analyzing inventory systems where there is considerable uncertainty about future demands. In this section, we consider a *continuous-review* inventory system. Thus, the inventory level is being monitored on a continuous basis so that a new order can be placed as soon as the inventory level drops to the reorder point.

The traditional method of implementing a *continuous-review* inventory system was to use a **two-bin system**. All the units for a particular product would be held in two bins. The capacity of one bin would equal the reorder point. The units would first be withdrawn from the other bin. Therefore, the emptying of this second bin would trigger placing a new order. During the lead time until this order is received, units would then be withdrawn from the first bin.

In more recent years, two-bin systems have been largely replaced by **computerized inventory systems**. Each addition to inventory and each sale causing a withdrawal are recorded electronically, so that the current inventory level always is in the computer. (For example, the modern scanning devices at retail store checkout stands may both itemize your purchases and record the sales of stable products for purposes of adjusting the current inventory levels.) Therefore, the computer will trigger a new order as soon as the inventory level has dropped to the reorder point. Several excellent software packages are available from software companies for implementing such a system.

Because of the extensive use of computers for modern inventory management, continuous-review inventory systems have become increasingly prevalent for products that are sufficiently important to warrant a formal inventory policy.

A continuous-review inventory system for a particular product normally will be based on two critical numbers:

R = reorder point.

Q = order quantity.

For a manufacturer managing its finished products inventory, the order will be for a *production run* of size Q . For a wholesaler or retailer (or a manufacturer replenishing its raw materials inventory from a supplier), the order will be a *purchase order* for Q units of the product.

An inventory policy based on these two critical numbers is a simple one. **Inventory policy:** Whenever the inventory level of the product drops to R units, place an order for Q more units to replenish the inventory.

Such a policy is often called a *reorder-point, order-quantity policy*, or **(R , Q) policy** for short. [Consequently, the overall model might be referred to as the (R, Q) model. Other variations of these names, such as (Q, R) policy, (Q, R) model, etc., also are sometimes used.]

After summarizing the model's assumptions, we will outline how R and Q can be determined.

The Assumptions of the Model

1. Each application involves a single product.
2. The inventory level is under *continuous review*, so its current value always is known.
3. An (R, Q) policy is to be used, so the only decisions to be made are to choose R and Q .
4. There is a *lead time* between when the order is placed and when the order quantity is received. This lead time can be either fixed or variable.
5. The *demand* for withdrawing units from inventory to sell them (or for

any other purpose) during this lead time is uncertain. However, the probability distribution of demand is known (or at least estimated).

6. If a stock-out occurs before the order is received, the excess demand is *backlogged*, so that the backorders are filled once the order arrives.
7. A fixed *setup cost* (denoted by K) is incurred each time an order is placed.
8. Except for this setup cost, the cost of the order is proportional to the order quantity Q .
9. A certain holding cost (denoted by h) is incurred for each unit in inventory per unit time.
10. When a stock-out occurs, a certain shortage cost (denoted by p) is incurred for each unit backordered per unit time until the backorder is filled.

This model is closely related to the *EOQ model with planned*. In fact, all these assumptions also are consistent with that model, with the one key exception of assumption 5. Rather than having uncertain demand, that model assumed *known demand* with a fixed rate.

Because of the close relationship between these two models, their results should be fairly similar. The main difference is that, because of the uncertain demand for the current model, some safety stock needs to be added when setting the reorder point to provide some cushion for having well-above-average demand during the lead time. Otherwise, the trade-offs between the various cost factors are basically the same, so the order quantities from the two models should be similar.

Choosing the Order Quantity Q

The most straightforward approach to choosing Q for the current model is to simply use the formula for the EOQ model with planned shortages. This formula is

$$Q = \sqrt{\frac{2AK}{h}} \sqrt{\frac{p+h}{p}}$$

where A now is the *average* demand per unit time, and where K , h , and p are defined in assumptions 7, 9, and 10, respectively.

This Q will be only an approximation of the optimal order quantity for the current model. However, no formula is available for the exact value of the optimal order quantity, so an approximation is needed. Fortunately, the approximation given above is a fairly good one.

1.6 A Stochastic Single-Period Model for Perishable Products

When choosing the inventory model to use for a particular product, a distinction should be made between two types of products. One type is a **stable product**, which will remain sellable indefinitely so there is no deadline for disposing of its inventory. This is the kind of product considered in the preceding sections (as well as the next section). The other type, by contrast, is a **perishable product**, which can be carried in inventory for only a very limited period of time before it can no longer be sold.

One example of a perishable product is a daily newspaper being sold at a newsstand. A particular day's newspaper can be carried in inventory for only a single day before it becomes outdated and needs to be replaced by the next day's newspaper. When the demand for the newspaper is a random variable, the owner of the newsstand needs to choose a daily order quantity that provides an appropriate trade-off between the potential cost of over ordering (the wasted expense of ordering more newspapers than can be sold) and the potential cost of under ordering (the lost profit from ordering fewer newspapers than can be sold). This section's model enables solving for the daily order quantity that would maximize the expected profit.

Because the general problem being analyzed fits this example so well, the problem has traditionally been called the **newsboy problem**. However, it has always been recognized that the model being used is just as applicable to other perishable products as to newspapers. In fact, most of the applications have been to perishable products other than newspapers, including the examples of perishable products listed below.

Some Types of Perishable Products

As you read through the list below of various types of perishable products, think about how the inventory management of such products is analogous to a newsstand dealing with a daily newspaper since these products also cannot be sold after a single time period. All that may differ is that the length of this time period may be a week, a month, or even several months rather than just one day.

1. Periodicals, such as newspapers and magazines.
2. Flowers being sold by a florist.
3. The makings of fresh food to be prepared in a restaurant.
4. Produce, including fresh fruits and vegetables, to be sold in a grocery store.
5. Christmas trees.
6. Seasonal clothing, such as winter coats, where any goods remaining at the end of the season must be sold at highly discounted prices to clear space for the next season.
7. Seasonal greeting cards.
8. Fashion goods that will be out of style soon.
9. New cars at the end of a model year.
10. Any product that will be obsolete soon.
11. Vital spare parts that must be produced during the last production run of a certain model of a product (e.g., an airplane) for use as needed throughout the lengthy field life of that model.
12. Reservations provided by an airline for a particular flight. Reservations provided in excess of the number of seats available (overbooking) can be viewed as the inventory of a perishable product (they cannot be sold after the flight has occurred), where the demand then is the number of no-shows. With this interpretation, the cost of under ordering (too little

overbooking) would be the lost profit from empty seats and the cost of over ordering (too much overbooking) would be the cost of compensating bumped customers.

1.7 Stochastic Periodic-Review Models

The preceding section presented a stochastic single-period model that is designed for dealing with *perishable* products. We now return to considering *stable* products that will remain sellable indefinitely, as in the first five sections of the chapter. We again assume that the demand is uncertain so that a stochastic model is needed. However, in contrast to the continuous-review inventory system we assume that the system is only being monitored periodically. At the end of each period, when the current inventory level is determined, a decision is made on how much to order (if any) to replenish inventory for the next period. Each of these decisions takes into account the planning for multiple periods into the future.

We begin with the simplest case where the planning is only being done for the next two periods and no setup cost is incurred when placing an order to replenish inventory.

A Stochastic Two-Period Model with No Setup Cost

One option with a stochastic periodic-review inventory system is to plan ahead only one period at a time, using the stochastic single-period model from the preceding section to make the ordering decision each time. However, this approach would only provide a relatively crude approximation. If the probability distribution of demand in each period can be forecasted multiple periods into the future, better decisions can be made by coordinating the plans for all these periods than by planning ahead just one period at a time. This can be quite difficult for many periods, but is considerably less difficult when considering only two periods at a time.

Even for a planning horizon of two periods, using the optimal one-period solution twice is not generally the optimal policy for the two-period problem. Smaller costs can usually be achieved by viewing the problem from a two-period viewpoint and then using the methods of probabilistic dynamic programming to obtain the best inventory policy.

Stochastic Multi-period Models - An Overview

The two-period model can be extended to several periods or to an infinite number of periods. This section presents a summary of multi-period results that have practical importance.

Multi-period Model with No Setup Cost.

Consider the direct extension of the above two-period model to n periods ($n > 2$) with the identical assumptions. The only difference is that a *discount factor* a with $0 < a < 1$, now will be used in calculating the expected total cost for n periods. The problem still is to find the critical numbers $y_1^0, y_2^0, \dots, y_n^0$ that describe the optimal inventory policy. As in the two period model, these values are difficult to obtain numerically, but it can be shown that the optimal policy has the following form.

For each period i ($i = 1, 2, \dots, n$), with x_i as the inventory level entering that period (before replenishing), do the following

$$\text{if } x_i \begin{cases} < y_i^0 & \text{order } y_i^0 - x_i \text{ to bring inventory level up to } y_i^0 \\ \geq y_i^0 & \text{do not order in period } i. \end{cases}$$

Furthermore,

$$y_1^0 \leq y_{n-1}^0 \leq \dots \leq y_2^0 \leq y_1^0$$

For the *infinite-period* case (where $n = \infty$), all these critical numbers y_1^0, y_2^0, \dots are *equal*. Let y^0 denote this constant value. It can be shown that y^0 satisfies the equation

$$\Phi(y^0) = \frac{p - c(1 - \alpha)}{p + h}$$

When the demand has either a uniform or an exponential distribution, an Excel template is available in your OR Courseware for calculating y^0 .

A Variation of the Multi-period Inventory Model with No Setup Cost. These results for the infinite-period case (all the critical numbers equal the same value y^0 and y^0

satisfies the above equation) also apply when n is finite if two new assumptions are made about what happens at the end of the last period. One new assumption is that each unit left over at the end of the final period can be salvaged with a return of the initial purchase cost c . Similarly, if there is a shortage at this time, assume that the shortage is met by an emergency shipment with the same unit purchase cost c .

1.8 Larger Inventory Systems in Practice

All the inventory models presented in this chapter have been concerned with the management of the inventory of a single product at a single geographical location. Such models provide the basic building blocks of scientific inventory management.

1.8.1 Multiproduct Inventory Systems

However, it is important to recognize that many inventory systems must deal simultaneously with many products, sometimes even hundreds or thousands of products. Furthermore, the inventory of each product often is dispersed geographically, perhaps even globally.

With multiple products, it commonly is possible to apply the appropriate single-product model to each of the products individually. However, companies may not bother to do this for the less important products because of the costs involved in regularly monitoring the inventory level to implement such a model. One popular approach in practice is the **ABC control method**. This involves dividing the products into three groups called the *A* group, *B* group, and *C* group. The products in the *A* group are the particularly important ones that are to be carefully monitored according to a formal inventory model. Products in the *C* group are the least important, so they are only monitored informally on a very occasional basis. Group *B* products receive an intermediate treatment.

It occasionally is not appropriate to apply a single-product inventory model because of interactions between the products. Various interactions are possible. Perhaps similar products can be substituted for each other as needed. For a manufacturer, perhaps its products must compete for production time when ordering production runs. For a wholesaler or retailer, perhaps its setup cost for ordering a product can be reduced by placing a joint order for a number of products simultaneously. Perhaps

there also are joint budget limitations involving all the products. Perhaps the products need to compete for limited storage space. It is common in practice to have a little bit of such interactions between products and still apply a single-product inventory model as a reasonable approximation. However, when an interaction is playing a major role, further analysis is needed. Some research has been conducted already to develop *multiproduct inventory models* to deal with some of these interactions.

1.8.2 Multi-echelon Inventory Systems

Echelon inventory is defined as the inventory between a stage in the supply chain and the final customer. This helps when we are determining the safety inventories at an individual stage in the supply chain because it will depend not on the local inventory but it will depend on the inventory that is available at the previous and the next stages too and will help in minimizing the safety inventories for the individual stages. The placement of inventory with multi-echelon supply chains is a major driver of cost and level of customer service.

The common situation will occur in two stages. In the first stage, products are distributed from the supplier to a central warehouse. In the second stage, products are re-distributed to different local DC's (clusters). The multi-echelon inventory uses in multi stages (Grouped) in a supply chain. The complexities of managing inventory increase significantly for a multi-echelon distribution network with multiple tiers of locations (e.g., a network comprising a central warehouse and downstream customer-facing locations). All locations are under the internal control of a single enterprise. Instead of simply replenishing the Warehouse or the DC's that sit between your supplier and your end customers, as in the single echelon situation, you also need to contend with the problems of replenishing another distribution point between your supplier and your DC's.

The objective of multi-echelon inventory management is to deliver the desired end customer service levels at minimum network inventory, with the inventory divided among the various echelons. The characteristics are Multiple lead times, Multiple stocking positions, and Often possibility to cross-dock.

Basic structures:**(a) A serial two-echelon system**

The simplest possible multi-echelon inventory system is one where there are only two echelons and only a single installation at each echelon.

(b) A serial multi-echelon system

A serial system with more than two echelons is known as multi-echelon system. The following figure depicts this kind of system, where installation 1 has its inventory replenished periodically, then the inventory at installation 1 is used to replenish the inventory at installation 2 periodically, then installation 2 does the same for installation 3, and so on down to the final installation (installation N)

Extended structures:**(a) Distribution system**

The two models presented previously in this section represent serial inventory system. Many real multi-echelon inventory systems are more complicated than this. An installation might have multiple immediate successors, such as when a factory supplies multiple Warehouses or when a Warehouse supplies multiple retailers; such an inventory system is called distribution system.

(b) Assembly system

Another common generalization of a serial multi-echelon inventory system arises when some installations have multiple predecessors, such as when a sub-assembly plant receives its components from multiple suppliers or when a factory receives its sub-assemblies from multiple sub-assembly plants. Such an inventory system is called an assembly system.

The role of inventory in SCM

The operational activities of the Supply Chain inventories include:

(1) Raw materials: inventory planning, purchasing, transportation from vendor to manufacturers and storage. (2) Work-in-process (WIP): processing and/or

preprocessing inside the manufacturing unit, (3) Finished goods: warehousing, finished product inventory management, delivery among the wholesalers and retailers. An effective control of an inventory system increases the production efficiency of a system.

1.8.3 Multi-echelon Inventory Management at IBM

IBM has roughly 1,000 products in service. Therefore, it employs over 15,000 customer engineers who are trained to repair and maintain all the installed computer systems sold or leased by IBM throughout the United States.

To support this effort, IBM maintains a huge multi-echelon inventory system of spare parts. This system controls over 200,000 part numbers, with the total inventory valued in the *billions of dollars*. Millions of parts transactions are processed annually.

The echelons of this system start with the manufacture of the parts, then national or regional warehouses, then field distribution centers, then parts stations, and finally many thousand outside locations (including customer stock locations and the car trunks or tool chests of the company's customer engineers).

To coordinate and control all these inventories at the different echelons, a huge computerized system called *Optimizer* was developed. Optimizer consists of four major modules. A forecasting system module contains a few programs for estimating the failure rates of individual types of parts. A data delivery system module consists of approximately 100 programs that process over 15 gigabytes of data to provide the needed input into Optimizer.

A decision system module then optimizes control of the inventories on a weekly basis. The fourth module includes six programs that integrate Optimizer into IBM's Parts Inventory Management System (PIMS). PIMS is a sophisticated information and control system that contains millions of lines of code.

Optimizer tracks the inventory level for each part number at all stocking locations (except at the outside locations, where only parts costing more than a certain threshold are tracked). An (R, Q) type of inventory policy is used for each part at each location and echelon in the system.

Careful planning was required to *implement* such a complex system after it had been designed. Three factors proved to be especially important in achieving a successful implementation. The first was the inclusion of a *user team* (consisting of operational managers) as advisers to the project team throughout the study. By the time of the implementation phase, these operational managers had a strong sense of ownership and so had become ardent supporters for installing Optimizer in their functional areas. A second success factor was a very extensive *user acceptance test* whereby users could identify problem areas that needed rectifying prior to full implementation. The third key was that the new system was phased in gradually, with careful testing at each phase, so the major bugs would be eliminated before the system went live nationally.

This new multi-echelon inventory system proved to be extremely successful. It provided savings of about \$20 million per year through improved operational efficiency. It also gave even larger annual savings in holding costs (including the cost of capital tied up in inventory) by reducing the value of IBM's inventories by over \$250 million. Despite this large reduction in inventories, the improved inventory management still enabled providing better service to IBM's customers. Specifically, the new system yielded a 10 percent improvement in the parts availability at the lower echelons (where the customers are affected) while maintaining the parts availability levels at the higher echelons.

1.8.4 Supply Chain Management

Another key concept that has emerged in this global economy is that of supply chain management. This concept pushes the management of a multi-echelon inventory system one step further by also considering what needs to happen to bring a product into the inventory system in the first place. However, as with inventory management, a main purpose still is to win the competitive battle against other companies in bringing the product to the customers as promptly as possible.

A **supply chain** is a network of facilities that procure raw materials, transform them into intermediate goods and then final products, and finally deliver the products to customers through a distribution system that includes a (probably multi-echelon) inventory system. Thus, it spans procurement, manufacturing, and distribution, with effective inventory management as one key element. To fill orders efficiently, it is

necessary to understand the linkages and interrelationships of all the key elements of the supply chain. Therefore, integrated management of the supply chain has become a key success factor for some of today's leading companies.

We summarize below the experience of one of the companies that have led the way in making supply chain management part of their corporate culture.

1.8.5 Supply Chain Management at Hewlett-Packard

Hewlett-Packard (HP) is one of today's leading high-technology companies. Its scope is truly global. Nearly half of its employees are outside the United States. In 1993, it had manufacturing or research and development sites in 16 countries, as well as sales and service offices in 110 countries. Its total number of catalog products exceeded 22,000.

Late in the 1980s, HP faced inventories mounting into the billions of dollars and alarming customer dissatisfaction with its order fulfillment process. Management was very concerned, since order fulfillment was becoming a major battlefield in the high-technology industries. Recognizing the need for OR models to support top management decision making, HP formed a group known as Strategic Planning and Modeling (SPaM) in 1988. Management charged the group with developing and introducing innovations in OR and industrial engineering.

In 1989, SPaM began bringing supply chain management concepts into HP. HP's supply chain includes manufacturing integrated circuits, board assembly, final assembly, and delivery to customers on a global basis. With such diverse and complex products, grappling with supply chain issues can be very challenging. Variability's and uncertainties are prevalent all along the chain. Suppliers can be late in their shipments, or the incoming materials may be flawed. The production process may break down, or the production yield may be imperfect. Finally, product demands also are highly uncertain. Much of SPaM's initial focus was on inventory modeling. This effort led to the development of HP's *Worldwide Inventory Network Optimizer (WINO)*. Like IBM's Optimizer described earlier in this section, WINO manages a multi-echelon inventory system.

However, rather than dealing just with inventories of finished products, WINO also considers the inventories of incoming goods and departing goods at each site along the supply chain.

WINO uses a discrete-review inventory model to determine the reorder point and order quantities for each of these inventories. By introducing more frequent reviews of inventories, better balancing of related inventories, elimination of redundant safety stocks, etc., inventory reductions of 10 to 30 percent typically were obtained.

WINO was even extended to include the inventory systems of some key dealers. This enabled reducing the inventories of finished products at both HP's distribution centers and the dealers while maintaining the same service target for the customers.

SPaM's initial focus on inventory modeling soon broadened to dealing with distribution strategy issues. For example, its realignment of the distribution network in Europe reduced the total distribution cost there by \$18 million per year.

SPaM's work also evolved into other functional areas, including design and engineering, finance, and marketing.

The importance of supply chain management now is recognized throughout HP. Several key divisions have formalized such positions as supply chain project managers, supply chain analysts, and supply chain coordinators. These individuals work closely with SPaM to ensure that supply chain models are used effectively and to identify new problems that feed SPaM's research and development effort.

The work of SPaM in applying OR to integrate supply chain management into HP has paid tremendous dividends. SPaM has often identified cost savings of \$10 million to \$40 million per year from just a single project. Therefore, total cost savings now run into the hundreds of millions of dollars annually. There have been key intangible benefits as well, including enhancing HP's reputation as a progressive company that can be counted on by its customers to fill their orders promptly.

1.9 Literature survey

Inventory control is the key to achieve enterprise cooperation and supply chain integrated management. Therefore, it is very meaningful in both theory and reality to research on inventory control and achieve optimization of supply chain. Clark and Scarf (1960) [13] put forward the multi-echelon inventory first. They analyzed an N-echelon pipelining system without considering a lot size, and proved that the optimal inventory control strategy of N-echelon pipelining system considering the shortage cost and inventory cost as the *optimal order level*. Recent developments in two-echelon models may be found in Qi, Ming.He and E.M. Jewkes(2000)[61]. Sven Axsäter (1990)[7] proposed an approximate model of inventory structure in SC. This model presumed that the demand of distribution centre and retailers obeyed the Normal distribution. This method reduced the algorithmic complexity and it was applicable for large-scale inventory. But when the individual retail demand is small, the normal distribution to approximate demand would have some negative value, and hence it is un-reasonable in some degree.

The first quantitative analysis in inventory studies started with the work of Harris (1915)[28]. He formulated mathematically, a simple inventory situation and obtained its solution. Wilson rediscovered the same formula in 1918. After the Second World War, several researchers like Pierre Masse (1946)[57], Arrow, Harris and Marschack (1951)[6], Dvoretzky, Keifer, and Wolfowitz (1952)[15] and Whittin (1953)[88], discussed the stochastic nature of inventory problems. One of the major operating policies in the study of stochastic inventory systems is the well established (s, S) policy. This policy has a simple operating rule: when the inventory position is less than or equal to a prefixed level s, an order is placed to raise it to the maximum level S. The probabilistic approach for such systems was proposed by Arrow, Karlin and Scarf (1958)[5], who provided a systematic treatment of inventory models based on renewal theory.

An efficient algorithm for computing optimal policies was designed by Federgruen and Zipkin (1984) [20] developed a conceptual framework for manufacturing strategy analysis, where they described a series of stochastic sub-models, that consider annualized product flows from raw material vendors via

intermediate plants and distribution echelons to the final customers. They used heuristic methods to link and optimize these sub- models. They later gave an integrated and readable exposition of their models and methods. They developed a normative model for resource deployment in a global manufacturing and distribution network. Global after-tax profit (profit-local taxes) was maximized through the design of facility network and control of material flows within the network. The cost structure consisted of variable and fixed costs for material procurement, production, distribution and transportation. They validated the model by applying it to analyze the global manufacturing strategies of a personal computer manufacturer.

Hadley and Whitin (1963)[27] provided an excellent account of inventory control theory and applications. A computational approach for finding optimal (s, S) inventory policies was given by Veinott and Wagner (1965)[86]. An excellent review by Veinott (1966)[85], summarized the status of mathematical theory of inventory until the early sixties. He focused his attention on the determination of optimal policies of multi - item and / or for multi-echelon inventory systems with certain and uncertain demands.

For continuous review systems, which may be considered as a limiting case of periodic review models, one can argue heuristically that the (s,S) policy is optimal. The rigorous proof was already available Beckmann (1987) [10], Hadley and Whitin(1963) [27] and Sahin(1979) [69]. In continuous review systems, the prime managerial level of interest is to determine the optimal values of s and S that minimize the long-run expected cost rate. The management practitioners expect a closed form solution for this problem, which is sometimes very difficult to obtain. The computational complexity of cost function is high, because it is a function of two variables s and S . A continuous review production inventory system was studied elaborately by Sharafali(1984)[76]. The cost analysis of different inventory systems along with several other characteristics is given in Naddor (1966)[48]. Gross and Harris (1973)[26], developed continuous review (s, S) inventory models with state dependent lead times, and they (1971)[25], dealt with an idea of dependence between replenishment times and the number of outstanding orders. Sivazlian (1974)[78], considered a continuous review (S, s) inventory system with arbitrary inter-arrival time distribution between demands, where each arrival demands exactly one unit. He

obtained the transient and steady state distribution for the position inventory and showed that the limiting distribution of the position inventory was uniform and was independent of the inter-arrival time distribution under many sharp assumptions. The same result for the case with arbitrarily distributed demand quantity was obtained by Richards (1975)[67]. Later (1978)[68], he dealt with a continuous review (s, S) inventory system in which the demand for items in inventory was dependent on an external environment.

An in-depth study of (s, S) inventory policy with arbitrarily distributed lead time is available in Srinivasan (1979)[79], and extended Sivazlian's result to the case of random lead times. He derived explicit expression for probability mass function of the stock level and extracted steady state results from the general formulae. This was further extended by Manoharan, Krishnamoorthy and Madhusoodanan (1987)[44], to the case of non-identically distributed inter-arrival demand times and random lead times, which, however, was restricted to demand quantity being exactly equal to one unit. Here he assumed the demand process as a renewal process where as Sahin (1983)[70], considered an inventory problem with the item being continuously measured; inter-arrival times formed a renewal process. However, he assumes the lead time to be a degenerate random variable. An inventory system (s, S) with demand for items dependent on an external environment was studied by Feldman (1975)[22]. Ramaswami (1981)[63] obtained algorithms for an (s, S) inventory model where the demand stood according to a versatile Markovian point process.

Studies of heuristic methods can be found in Wagner(1962) [87], Naddor(1966)[48], Erharat and Wagner(1982) [19], Porteus(1971) [58], Sahin and Sinha [71]. Because of the ill behavior of the cost function, the computation of the optimal values for s and S was found to be prohibitively expensive. Success stories of applications of (s, S) inventory models can be found in Nahmias(1982)[49]. Kalpakam and Sapna (1993a)[29] analyzed an (s, S) ordering policy in which items were procured on an emergency basis during stock-out period. Kalpakam and Sapna (1993b)[30] dealt with the problem of controlling the replenishment rates in a lost sales inventory system with compound Poisson demands and two types of reorders with varying order quantities. Prasad (1994) [60] developed a new classification system that compared different inventory systems.

A modified (s, S) inventory system with lost sales was studied by Kalpakam and Sapna (1997) [32]. They considered a system having renewal demands and exponential lead times, with a policy involving placement of an emergency order at the time of stock-out, if available. But all the models cited above deal with lost sales and restricted reorder policy that the number of pending orders at any instant be unity ($S-s > s$) to preserve the tractability of the analysis.

Archibald (1981) [2] analyzed a lost sales continuous review (s, S) system and derived the long run expected cost rate for the same. A generalized study of lost sales (s, S) inventory system was done by Srinivasan. Kalpakam et.al. studied more models with lost sales. Kalpakam and Arivarignan introduced multiple reorder level policy with lost sales in inventory control system. Elango and Arivarignan (2001) [16] considered an inventory model in which they introduced an ordering policy with contiguous collection of reorder levels and lost sales. The demand points formed a renewal process and the lead time was distributed as negative exponential. Perumal et.al. (2003) [56] considered a Markovian inventory system with a set of reorder levels but the lead time was assumed to be distributed as exponential with parameter depending upon the level at which it was initiated.

Kalpakam and Arivarignan (1998) [35] introduced perishability of exhibiting item(s) and provided several characterization of the underlying inventory process. They (1985a) [36] considered the case of an inventory system with arbitrary inter arrival time between demands in which one item was put into operation as an exhibiting item whose lifetime has the exponential distribution. It was also assumed that non-exhibited items did not deteriorate. The transient and steady state distributions for position inventory were derived under the assumption that quantity demanded at a demand epoch depends the time elapsed since the previous demand point. Again the same system having one exhibiting item subject to random failures with failure times following exponential distribution and unit demand was dealt by the same authors (1985b) [37] and the expression for the limiting distribution of the position inventory was derived by applying the techniques of semi-regenerative process. The Markovian (s, S) inventory system with lost sales exponential lifetimes and zero lead times were studied by Kalpakam and Arivarignan. A continuous review Perishable inventory system at Service Facilities was studied by Elango (2001) [18].

The analysis of perishable systems with zero lead times is relatively straight forward, but if the lead time is positive, this becomes difficult as the decay applies not only to items on hand but also to those on-transit. The first paper that dealt analytically a model with positive lead times was that of Schmidt and Nahmias (1985)[74], in which it was assumed a $(S-1, S)$ policy, lost sales model with fixed life and lead times under Poisson demand. Ravichandran (1984)[64] dealt with (s, S) policy inventory system where the batch of $(S-s)$ items replenished would perish together and the lifetime had an Erlangian distribution. Pal (1990)[55].

Multi-echelon inventory system has been studied by many researchers and its applications in supply chain management has proved worthy in recent literature. As supply chains integrate many operators in the network and optimize the total cost involved without compromising customer service efficiency. A continuous review inventory system in SCM was studied by Krishnan.K (2007) [43].

One of the oldest papers in the field of continuous review multi-echelon inventory system was a basic and seminal paper written by Sherbrooke(1968) [77] . He assumed $(S-1, S)$ policies in the Depot-Base systems for repairable items in the American Air Force and could approximate the average inventory and stock out level in bases.

Continuous review models of multi-echelon inventory system in 1980's concentrated more on repairable items in a Depot-Base system than as consumable items (see Graves(1982) [23], Moinzadeh and Lee(1986) [47]). All these papers dealt with repairable items with batch ordering.

The research on consumable items in multi-echelon started with the first work by Deuermeyer and Schwarz(1981) [14]. He used a simple approximation for a complex multi-echelon (one warehouse and multiple retailers) assuming the full backordering in all installations with batch ordering policies. Svoronos, Antony and Zipkin(1991) [82] proposed several refinements by considering second-moments information (standard deviation and mean) in their approximation.

In all the above models, either one-for-one policy i.e. $(S, S-1)$ or batch ordering policy was considered for the system which simplified the two-echelon

system performance computations. In this thesis it is assumed (s, S) policy at retailers and $(0, M)$ policy at Distribution center.

Since this thesis provides results on retrial inventory. Inventory with postponed demands and inventory with service times we first give the motivation for considering such results.

From Retrial Queues to Retrial Inventory

Queueing system in which arriving customers who find all servers and waiting position (if any) occupied, may retry for service after a period of time, are called retrial queues or queues with repeated attempts. The most obvious example is provided by a person who desires to make a phone call. If the line is busy, then he can not queue up but tries again some time later. Thus, retrial queues are characterised by the following feature: a customer arriving when all servers accessible for him are busy, leaves the service area but after some random time repeats his demand. Retrial queues are a type of network with reserivicing after blocking.

Thus, this network contains two nodes: the main node where blocking is possible and a delay node for repeated attempts. As for other networks with blocking, the investigation of such systems presents great analytical difficulties. Nevertheless, the main feature of the theory of retrial queueing systems as an independent part of queueing theory are quite clearly drawn. In particular, the nature of results obtained, methods of analysis and areas of applications allow us to divide retrial queues into three large groups in a natural way: Single-channel system, multi-channel fully available systems and structurally complex systems. The standard queueing models do not take into account the phenomenon of retrials and therefore can not be applied in solving a number of practically important problems. Retrial queues have been introduced to solve this deficiency.

On the other hand retrial in inventory occurs as follows: Customers arrive to an establishment demanding an item. If the item is available the same is supplied (may be with negligible service time or with a positive (not necessarily random) service time). However, when at a demand epoch the item is out of stock, such customers need not be backlogged nor lost. An alternative to these is the retrial by

such customers. At random epochs of time such customer retries until either the demand is met or finally the customer decides not to approach that establishment (may be he is no more in need of the item or he procures it from elsewhere).

Queues with Postponed Work and Inventory with Pooled Customers

Postponement of work is a common phenomena. This may be to attend a more important job than the one being processed at present or for a break or due to lack of quorum (in case of bulk service) and so on. Postponement of service to customers take place in different ways depending on the nature of the input and service process. For example in the case of priority queues service to customers of lower priority stands postponed when one of the higher priority calls on. In the case of preemptive service, customers of lower priority in service is pushed out the moment one with higher priority arrives. In control policies such as N.T.D. a busy cycle stands only an accumulation of N customers in the system, an elapse of T time unit, the place or the work load accumulator to D , respectively. Hence these control policies can also be regarded as postponement of service.

On certain occasions postponement of work reduces partly or some time completely customer impatience, especially in the context of priority queues. There are several other means of reducing customer impatience. Of these the one introduced by Qi-Ming He[61] deserves special mention. They devised a control mechanism of a system consisting of two queues served by two different servers, by introducing transfer of customers in bulk from the larger to the shorter queue. They established that even when the queues are not separately stable the combined system can be stable. By identifying a two dimensional Markov chain . with one component representing the sum of the number of customers in the two queues and the other, the difference between queue 1 and queue 2, they analyzed the resulting system as a level independent QBD.

In this thesis we introduce postponement of supply of the items to a demand as described below. At a demand epoch if the item is out of stock then such customers are directed to a pool. Such customers are referred to as pooled customers / postponed work (demands). On replenishment customers from the pool are selected for providing the item according to some rules as described in chapters to follow. This is

an alternative to backlogging of demands where at the time of arrival of a customer the system is found to be out of stock. Whereas in backlogged case such customers are provided the item immediately on replenishment, in the case postponed demand, this facility is not extended to the customers. In the latter the system management takes the decisions as to when the postponed customers be served.

Inventory with Service Time

In all works reported in inventory prior to 1993 it was assumed that the time required to serve the item to the customer is negligible. Berman, Kim and Shimshak [9] is the first attempt to introduce positive service time in inventory, where it was assumed that service time is deterministic. Assuming Poisson arrival process, arbitrarily distributed service times and zero lead time they analyze the system with the restriction that waiting space is finite. Under a specific cost structure they derive the optimum ordering quantity that minimizes the long run expected cost rate. With all these still there are only a handful of papers that deals with inventory involving service time. In a few chapters to follow we consider inventory with random service times.

1.10 Outline of the Present Work

This thesis is divided into seven chapters including this introductory chapter.

Chapter-1 presents, some preliminary concepts in Inventory System and its developments, importance, different type of decision-making, and Multi-echelon inventory system. Besides, a brief review of the literature relevant to the topic. An outline of the work done in the present thesis are also discussed.

Chapter-2, some basic definitions and important concepts of Stochastic processes and their applications are given.

Chapter-3, deals with an integrated inventory control system that consists of one manufacturer (MF), one Distribution centre (DC), and one retailer. An (s,S) type inventory system with Poisson demand and exponentially distributed lead times are assumed at retailer and DC respectively. The demands occurring during the stock out period and the unsatisfied customer demand are permitted to enter into an orbit of size

N . Such unsatisfied customer demand is known as Retrial demand. The above network model is studied for both Non-perishable and Perishable products. The steady state probability distribution of system states and the measures of system performance are obtained for both the models. Numerical examples with sensitivity analysis are provided to illustrate the proposed models.

Chapter-4 deals a continuous review inventory control system consists of a Manufacturer (MF), single Warehouse (WH), one Distribution Centre (DC) and n identical retailers system handling a single perishable product. A (s, Q) type inventory system with Poisson demand and exponentially distributed lead times for items are assumed at DC (middle echelon). An one-for-one type inventory policy is assumed at retailers node (lower echelon). Demands occurring during the stock out periods are assumed to be lost. The WH (upper echelon) replenishes their stocks from Manufacturer, which has abundant stocks for supply. The items are supplied to the Manufacturer in packs of Q items from the warehouse. The perishable product has the exponential decay of constant rate. Here the exposure of items occurs only at retailer node. The steady state probability distribution and the operating characteristics are obtained explicitly. The required algorithm is designed and it is executed.

Chapter-5, deals with a simple supply chain that is modeled as a single warehouse and multiple retailer system handling a single product. In order to avoid the complexity, at the same time without loss of generality, we assumed identical demand pattern at each node. This restricts our study to design and analyze a tandem network of inventory, which is a building block for the whole supply chain system. A (s, S) type inventory system with Poisson demand and exponentially distributed lead times is assumed at retailer node. The items are supplied to the retailers and the direct customers (Direct demand), in packs of $Q (= S-s)$ items from the distribution center which has instantaneous replenishment facility from an abundant source (manufacturer). The unsatisfied customers at retailer are entered into the orbit of finite capacity N . These orbiting demands retry for their demand after a random time which is assumed to be exponential distribution. The transient and steady state probability distribution and the operating characteristics are obtained explicitly. The measures of system performance in the steady state are obtained. The required algorithm is designed and it is executed.

Chapter-6, deals with a continuous review inventory system in supply chain (consist of retailer, distributor and manufacturer) with Markovian demand. The operating policy is (s, S) policy, that is, the maximum inventory level is S and whenever the inventory level drops to, s an order for $Q(S-s)$ units is placed. The items are supplied to the retailers and the direct customers (Direct demand), in packs of Q items from the distribution center which has instantaneous replenishment facility from an abundant source (manufacturer). The ordered items are received after a random time which is assumed to be exponential distribution. The demands that occur during stock out period are enter into the orbit of finite size N . These orbiting demands retry for their demand after a random time, which is assumed to be exponential distribution. The joint probability distribution of the inventory levels of both retailer and distributor and the number of customers in the orbit is obtained in the steady state case. Various system performance measures in the steady state are derived and the long-run total expected cost rate is calculated. Several instances of a numerical example, which provide insight into the behavior of the system, are presented.

In the last **Chapter-7**, the summary of the study, conclusions and scope for further work are given.

Thus chapter 1 has presented preliminary concepts in Inventory system, review of literature and the chapter wise development of the present study. The second chapter describes the concepts and definitions of stochastic processes and their applications. In the remaining chapters 3 to 6, we discussed various stochastic inventory models which is the body of the thesis. We conclude the thesis in chapter 7 with some future directions.

CHAPTER 2

STOCHASTIC PROCESSES

2.1 Introduction

2.1.1 Definition

Let T be a subset of $[0, \infty)$. A family of random variables $\{X_t\}_{t \in T}$, indexed by T , is called a stochastic (or random) process. When $T = \mathbb{N}$ (or $T = \mathbb{N}_0$), $\{X_t\}_{t \in T}$ is said to be a discrete-time process, and when $T = [0, \infty)$, it is called a continuous-time process.

When T is a singleton (say $T = \{1\}$), the process $\{X(t) \ t \in T\} \equiv X_1$ is really just a single random variable. When T is finite (e.g., $T = \{1, 2, \dots, n\}$), we get a random vector. Therefore, stochastic processes are generalizations of random vectors. The interpretation is, however, somewhat different. While the components of a random vector usually (not always) stand for different spatial coordinates, the index $t \in T$ is more often than not interpreted as time. Stochastic processes usually model the evolution of a random system in time. When $T = [0, \infty)$ (continuous-time processes), the value of the process can change every instant. When $T = \mathbb{N}$ (discrete-time processes), the changes occur discretely.

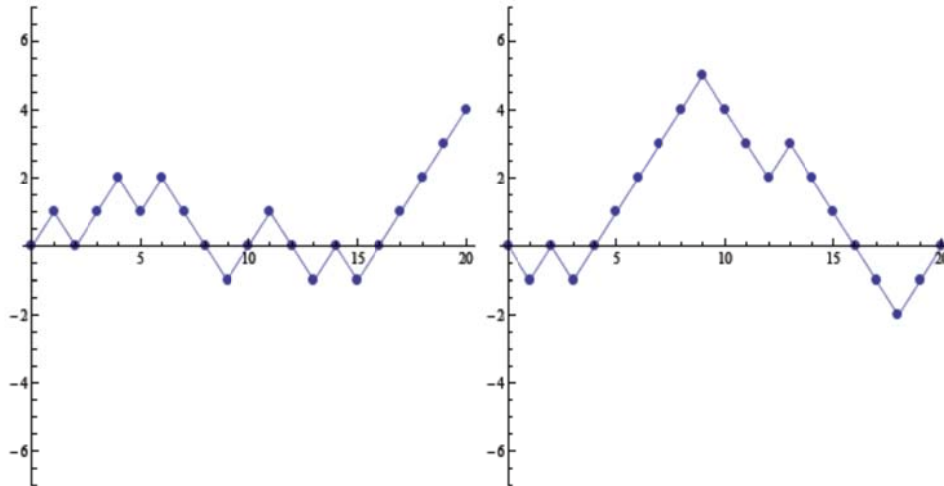
In contrast to the case of random vectors or random variables, it is not easy to define a notion of a density (or a probability mass function) for a stochastic process. Without going into details why exactly this is a problem, let me just mention that the main culprit is the infinity. One usually considers a family of (discrete, continuous, etc.) finite-dimensional distributions, i.e., the joint distributions of random vectors

$$(X_{t_1}, X_{t_2}, \dots, X_{t_n}),$$

for all $n \in \mathbb{N}$ and all choices $t_1, \dots, t_n \in T$.

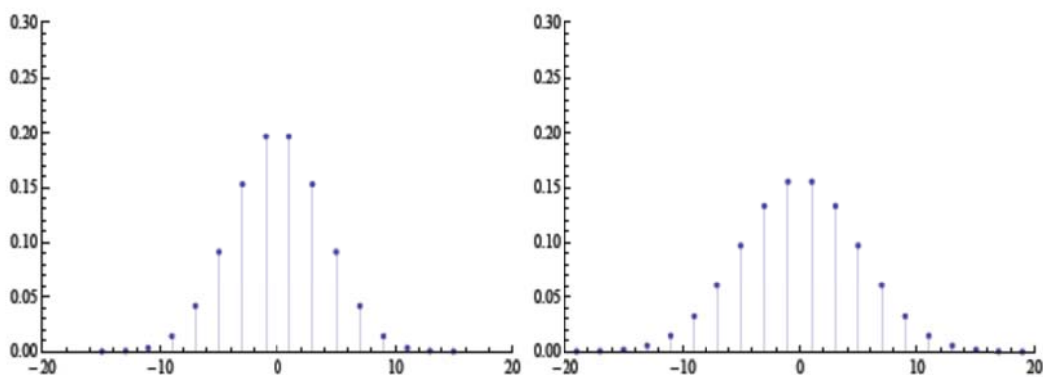
The notion of a stochastic processes is very important both in mathematical theory and its applications in science, engineering, economics, etc. It is used to model a large number of various phenomena where the quantity of interest varies discretely or continuously through time in a non-predictable fashion.

Every stochastic process can be viewed as a function of two variables - t and ω . For each fixed t , $\omega \mapsto X_t(\omega)$ is a random variable, as postulated in the definition. However, if we change our point of view and keep ω fixed, we see that the stochastic process is a function mapping ω to the real-valued function $t \mapsto X_t(\omega)$. These functions are called the trajectories of the stochastic process X .



Figures on the left show two different trajectories of a simple random walk, i.e., each one corresponds to a (different) frozen $\omega \in \Omega$, but t goes from 0 to 30.

We will define the simple random walk later. For now, let us just say that it behaves as follows. It starts at $x = 0$ for $t = 0$. After that a fair coin is tossed and we move up (to $x = 1$) if heads is observed and down to $x = -1$ if we see tails. The procedure is repeated at $t = 1, 2, \dots$ and the position at $t + 1$ is determined in the same way, independently of all the coin tosses before (note that the position at $t = k$ can be any of the following $x = -k, x = -k + 2, \dots, x = k - 2, x = k$).



Unlike with the figures above, the two pictures on the right show two time-slices of the same random process; in each graph, the time t is fixed ($t = 15$ vs. $t = 25$) but the various values random variables X_{15} and X_{25} can take are presented through the probability mass functions.

2.2 Markov Chains

A Markov process $\{X_s\}$ is a stochastic process with the property that, given the value of X the values of X , for $s > t$ are not influenced by the values of X for $u < t$. In words, the probability of any particular future behavior of the process, when its current state is known exactly, is not altered by additional knowledge concerning its past behavior. A discrete-time Markov chain is a Markov process whose state space is a finite or countable set, and whose (time) index set is $T = (0, 1, 2, \dots)$. In formal terms, the Markov property is that

$$\Pr\{X_{n+1} = j / X_0 = i_0, \dots, X_{n-1} = i\} = \Pr\{X_{n+1} = j / X_n = i\} \quad (1.1)$$

for all time points n and all states $i_0, \dots, i_{n-1}, i, j$.

It is frequently convenient to label the state space of the Markov chain by the nonnegative integers $\{0, 1, 2, \dots\}$, which we will do unless the contrary is explicitly stated, and it is customary to speak of X_n , as being in state i if $X_n = i$.

The probability of X_{n+1} being in state j given that X_n is in state i is called the one-step transition probability and is denoted by $P_{ij}^{n,n+1}$. That is,

$$P_{ij}^{n,n+1} = \Pr\{X_{n+1} = j / X_n = i\} \quad (1.2)$$

The notation emphasizes that in general the transition probabilities are functions not only of the initial and final states, but also of the time of transition as well. When the one-step transition probabilities are independent of the time variable n , we say that the Markov chain has stationary transition probabilities. Since the vast majority of Markov chains that we shall encounter have stationary transition probabilities, we limit our discussion to this case. Then $P_{ij}^{n,n+1} = P_{ij}$ is independent of n , and P_{ij} is the conditional probability that the state value undergoes a transition from i to j in one

trial. It is customary to arrange these numbers P_{ij} in a matrix, in the infinite square array

$$p = \begin{pmatrix} p_{00} & p_{01} & p_{02} & p_{03} & \dots \\ p_{10} & p_{11} & p_{12} & p_{13} & \dots \\ p_{20} & p_{21} & p_{22} & p_{23} & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \\ p_{i0} & p_{i1} & p_{i2} & p_{i3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix}$$

and refer to $p = \|p_{ij}\|$ as the Markov matrix or transition probability matrix of the process.

The i^{th} row of P , for $i = 0, 1, \dots$, is the probability distribution of the values of under the condition that $X_{n-1} = i$. If the number of states is finite, then P is a finite square matrix whose order (the number of rows) is equal to the number of states. Clearly, the quantities P_{ij} satisfy the conditions

$$p_{ij} \geq 0 \quad \text{for } i, j = 0, 1, 2, \dots \quad (1.3)$$

$$\sum_{j=0}^{\infty} p_{ij} = 1 \quad \text{for } i = 0, 1, 2, \dots \quad (1.4)$$

The condition (1.4) merely expresses the fact that some transition occurs at each trial. (For convenience, one says that a transition has occurred even if the state remains unchanged.) A Markov process is completely defined once its transition probability matrix and initial state X_0 (or, more generally, the probability distribution of X_0) are specified. We shall now prove this fact.

Definitions

Let $\Pr\{X_0 = i\} = p_i$. It is enough to show how to compute the quantities

$$\Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n\} \quad (1.5)$$

since any probability involving X_{j_1}, \dots, X_{j_K} , for $j_1 < \dots < j_K$, can be obtained, according to the axiom of total probability, by summing terms of the form (1.5).

By the definition of conditional probabilities we obtain

$$\begin{aligned} & \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n\} \\ &= \Pr\{X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}\} \end{aligned} \quad (1.6)$$

$$\times \Pr\{X_n = i_n / X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}\}.$$

.Now, by the definition of a Markov process,

$$\begin{aligned} & \Pr\{X_n = i_n / X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}\} \\ &= \Pr\{X_n = i_n / X_{n-1} = i_{n-1}\} = p_{i_{n-1}, j_n} \end{aligned} \quad (1.7)$$

Substituting (1.7) into (1.6) gives

$$\begin{aligned} & \Pr\{X_0 = i_0, X_1 = i_1, \dots, X_n = i_n\} = \Pr\{X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}\} = \\ & p_{i_{n-1}, j_n} \end{aligned}$$

Then, upon repeating the argument $n - 1$ additional times, (1.5) becomes

$$\Pr\{X_0 = i_0, X_1 = i_1, \dots, X_n = i_n\} = p_{i_0} p_{i_0, j_1} \dots p_{i_{n-2}, j_{n-1}} p_{i_{n-1}, j_n} \quad (1.8)$$

This shows that all finite-dimensional probabilities are specified once the transition probabilities and initial distribution are given, and in this sense the process is defined by these quantities. Related computations show that (1.1) is equivalent to the Markov property in the form

$$\begin{aligned} & p_r\{X_{n+1} = j_1, \dots, X_{n+m} = j_m / X_0 = i_0, \dots, X_n = i_n\} \\ &= p_r\{X_{n+1} = j_1, \dots, X_{n+m} = j_m / X_n = i_n\} \end{aligned} \quad (1.9)$$

for all time points n, m and all states $i_0, \dots, i_n, j_1, \dots, j_m$. In other words, once (1.9) is established for the value $m = 1$, it holds for all $m \geq 1$ as well.

2.3 Transition Probability Matrices of a Markov Chain

A Markov chain is completely defined by its one-step transition probability matrix and the specification of a probability distribution on the state of the process at time 0. The analysis of a Markov chain concerns mainly the calculation of the probabilities of the possible realizations of the process. Central in these calculations are the n-step transition probability matrices $p^{(n)} = \|p_{ij}^{(n)}\|$. Here p_{ij} denotes the probability that the process goes from state i to state j in n transitions. Formally,

$$p_{ij}^{(n)} = pr\{X_{m+n} = j / X_m = i\} \quad (2.1)$$

Observe that we are dealing only with temporally homogeneous processes having stationary transition probabilities, since otherwise the left side of (2.1) would also depend on m . The Markov property allows us to express (2.1) in terms of JJP I as stated in the following theorem.

Theorem 2.1

The n-step transition probabilities of a Markov chain satisfy

$$p_{ij}^{(n)} = \sum_{k=0}^{\infty} p_{ik} p_{kj}^{(n-1)} \quad (2.2)$$

where we define

$$p_{ij}^{(0)} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

From the theory of matrices we recognize the relation (2.2) as the formula for matrix multiplication, so that $p^{(n)} = p \times p^{(n-1)}$. By iterating this formula, we obtain

$$p^{(n)} = p \times p \times \dots \times p = p^n; \quad (2.3)$$

in other words, the n-step transition probabilities $P_{ij}^{(n)}$ are the entries in the matrix P^n , the n th power of P .

Proof

The proof proceeds via a first step analysis, a breaking down, or analysis, of the possible transitions on the first step, followed by an application of the Markov property. The event of going from state i to state j in n transitions can be realized in

the mutually exclusive ways of going to some intermediate state k ($k = 0, 1, \dots$) in the first transition, and then going from state k to state j in the remaining $(n - 1)$ transitions. Because of the Markov property, the probability of the second transition is $p_{kj}^{(n-1)}$ and that of the first is clearly P_{ik} . If we use the law of total probability, then (2.2) follows. The steps are

$$\begin{aligned} p_{ij}^{(n)} &= pr\{X_n = j / X_0 = i\} = \sum_{k=0}^{\infty} pr\{X_n = j, X_n = k / X_0 = i\} \\ &= \sum_{k=0}^{\infty} pr\{X_n = k / X_0 = i\} pr\{X_n = j / X_0 = i, X_1 = k\} \\ &= \sum_{k=0}^{\infty} P_{ik} P_{kj}^{(n-1)} \end{aligned}$$

If the probability of the process initially being in state j is p_j , i.e., the distribution law of X_0 is $Pr\{X_0 = j\} = p_j$, then the probability of the process being in state k at time n is

$$p_k^{(n)} = \sum_{j=0}^{\infty} p_j P_{kj}^{(n)} = pr\{X_n = k\}. \tag{2.4}$$

2.3.1 Classification of States

Not all Markov chains are regular. We consider some examples. The Markov chain whose transition probability matrix is the identity matrix

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

remains always in the state in which it starts. Since trivially $P^n = P$ for all n , the Markov chain X_n , has a limiting distribution, but it obviously depends on the initial state.

The Markov chain whose transition probability matrix is

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Oscillates deterministically between the two states. The Markov chain is periodic, and no limiting distribution exists. When n is an odd number, then $P^n = P$, but when n is even, then P^n is the 2×2 identity matrix. When P is the matrix

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

P^n is given by

$$P^n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos \theta^n & \sin \theta^n \\ -\sin \theta^n & \cos \theta^n \end{pmatrix},$$

and the limit is

$$\lim_{n \rightarrow \infty} P^n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Here state 0 is transient; after the process starts from state 0, there is a positive probability that it will never return to that state. The three matrices just presented illustrated three distinct types of behavior in addition to the convergence exemplified by a regular Markov chain. Various and more elaborate combinations of these behaviors are also possible. Some definitions and classifications of states and matrices are needed in order to sort out the variety of possibilities.

2.3.2. Irreducible Markov Chains

Since j is said to be accessible from state i if $p_{ij}^{(n)} > 0$ for some integer $n \geq 0$; i.e., state j is accessible from state i if there is positive probability that state j can be reached starting from state i in some finite number of transitions. Two states i and j , each accessible to the other, are said to communicate, and we write $i \leftrightarrow j$. If two states i and j do not communicate, then either

$$p_{ij}^{(n)} = 0 \quad \text{for all } n \geq 0$$

or both relations are true. The concept of communication is an equivalence relation:

(i) $i \leftrightarrow j$ (reflexivity), a consequence of the definition of

$$p_{ij}^{(0)} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

- (ii) If $i \leftrightarrow j$, then $j \leftrightarrow i$ (symmetry), from the definition of communication.
- (iii) If $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$ (transitivity).

The proof of transitivity proceeds as follows: $i \leftrightarrow j$ and $j \leftrightarrow k$ imply that there exist integers n and m such that $p_{ij}^{(n)} > 0$ and $p_{jk}^{(m)} > 0$. Consequently, by the nonnegativity of each $p_{ij}^{(n)}$ we conclude that

$$p_{ik}^{(n+m)} = \sum_{j=0}^{\infty} p_{ij}^{(n)} p_{jk}^{(m)} \geq p_{ij}^{(n)} p_{jk}^{(m)} > 0$$

A similar argument shows the existence of an integer v such that $p_{ki}^{(v)} > 0$, as desired. We can now partition the totality of states into equivalence classes. The states in an equivalence class are those that communicate with each other. It may be possible starting in one class to enter some other class with positive probability; if so, however, it is clearly not possible to return to the initial class, or else the two classes would together form a single class. We say that the Markov chain is irreducible if the equivalence relation induces only one class. In other words, a process is irreducible if all states communicate with each other. To illustrate this concept, we consider the transition probability matrix

$$\mathbf{P} = \left\| \begin{array}{cc|cc} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \end{array} \right\| = \left\| \begin{array}{c|c} \mathbf{P}_1 & \mathbf{0.0} \\ \mathbf{0} & \mathbf{P}_2 \end{array} \right\|, \tag{3.1}$$

where P , is an abbreviation for the matrix formed from the initial two rows and columns of P , and similarly for P_2 . This Markov chain clearly divides into the two classes composed of states $(1, 2)$ and states $\{3,4,5\}$.

If the state of X_0 lies in the first class, then the state of the system thereafter remains in this class, and for all purposes the relevant transition matrix is P . Similarly, if the initial state belongs to the second class, then the relevant transition matrix is R . This is a situation where we have two completely unrelated processes labeled together. In the random walk model with transition matrix states

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ q & 0 & p & 0 & \dots & 0 & 0 & 0 \\ 0 & q & 0 & p & \dots & 0 & 0 & 0 \\ \vdots & & & & & \vdots & \vdots & \vdots \\ 0 & & \dots & q & 0 & p & & \\ 0 & & \dots & 0 & 0 & 1 & & \end{pmatrix} \begin{matrix} 0 \\ 1 \\ 2 \\ \vdots \\ a-1 \\ a \end{matrix} \tag{3.2}$$

we have the three classes $(0), 1, 2, \dots, a - 1$, and $\{a\}$. In this example it is possible to reach the first class or third class from the second class, but it is not possible to return to the second class from either the first or the third class.

2.3.3 Periodicity of a Markov Chain

We define the period of state i , written $d(i)$, to be the greatest common divisor (g.c.d.) of all integers $n \geq 1$ for which $p^{(n)}_{ii} > 0$. [If $p^{(n)}_{ii} = 0$ for all $n \geq 1$, define $d(i) = 0$.] In a random walk (3.1), every transient state $1, 2, \dots, N - 1$ has period 2. If $P_{ij} > 0$ for some single state i , then that state now has period 1, since the system can remain in this state any length of time. In a finite Markov chain of n states with transition matrix

$$\mathbf{P} = \begin{pmatrix} \overbrace{0 & 1 & 0 & 0 & \dots & 0}^n \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & & \dots & 1 & \\ 1 & 0 & 0 & \dots & 0 & \end{pmatrix},$$

each state has period n . Consider the Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix} \end{matrix},$$

We evaluate $p_{00} = 0, p_{00}^{(2)} = 0, p_{00}^{(3)} = 0, p_{00}^{(4)} = \frac{1}{2}, p_{00}^{(5)} = 0, p_{00}^{(6)} = \frac{1}{4}$. The set of integers $n \geq 1$ for which $P > 0$ is $\{4, 6, 8, \dots\}$. The period of state 0 is $d(0) = 2$, the greatest common divisor of this set.

2.3.4 Birth and Death Process

$\Pr\{\text{Number of births between } t \text{ and } t+h \text{ is } k, \text{ given that the number of individuals at epoch } t \text{ is } n\}$

is given by

$$p(k, h|n, t) = \begin{cases} \lambda_n h + o(h), & k=1 \\ o(h), & k \geq 2 \\ 1 - \lambda_n h + o(h), & k=0 \end{cases} \quad (4.1)$$

The above holds for all $n \geq 0$; λ_0 may or may not be equal to zero. Here k is a non-negative integer which implies that there can only be an increase by k , i.e. only births are considered possible. Now we suppose that there could also be a decrease by k , i.e. death(s) is also considered possible. In this case we shall further assume that

$\Pr\{\text{Number of deaths between } t \text{ and } t+h \text{ is } k, \text{ given that the number of individuals at epoch } t \text{ is } n\}$ is given by

$$q(k, h|n, t) = \begin{cases} \mu_n h + o(h), & k=1 \\ o(h), & k \geq 2 \\ 1 - \mu_n h + o(h), & k=0 \end{cases} \quad (4.2)$$

The above holds for $n \geq k$; further $\mu_0 = 0$. With (4.1) and (4.2) we have, what is known as a birth and death process. Through a birth there is an increase by one and through a death, there is a decrease by one in the number of “individuals”. The probability of more than one birth or more than one death in an interval of length h is $o(h)$. Let $N(t)$ denotes the total number of individuals at epoch t starting from $t=0$ and let $p_n(t) = \Pr\{N(t)=n\}$. Consider the interval between 0 and $t+h$; suppose that it is split into two periods $(0, t)$ and $[t, t+h]$. The event $\{N(t+h)=n, n \geq 1\}$, (having the probability $p_n(t+h)$) can occur in a number of mutually exclusive ways.

These would include events involving more than one birth and/or more than one death between t and $t+h$. By our assumption, the probability of such an event is $o(h)$. There will remain four other events to be considered:

A_{ij} : $(n-i+j)$ individuals by epoch t , i birth and j death between t and $t+h$, $i, j=0, 1$.

We have

$$\begin{aligned} \Pr(A_{00}) &= p_n(t) \{1 - \lambda_n h + o(h)\} \{1 - \mu_n h + o(h)\} \\ &= p_n(t) \{1 - (\lambda_n + \mu_n) h + o(h)\}; \\ \Pr(A_{10}) &= p_{n-1}(t) \{\lambda_{n-1} h + o(h)\} \{1 - \mu_{n-1} h + o(h)\} \\ &= p_{n-1}(t) \lambda_{n-1} h + o(h); \\ \Pr(A_{01}) &= p_{n-1}(t) \{1 - \lambda_{n+1} h + o(h)\} \{\mu_{n+1} h + o(h)\} \\ &= p_{n+1}(t) \mu_{n+1} h + o(h); \end{aligned}$$

and

$$\begin{aligned} \Pr(A_{11}) &= p_n(t) \{\lambda_n h + o(h)\} \{\mu_n h + o(h)\} \\ &= o(h) \end{aligned}$$

Hence we have, for $n \geq 1$

$$p_n(t+h) = p_n(t) \{1 - (\lambda_n + \mu_n)h\} + p_{n-1}(t)\lambda_{n-1}h + p_{n+1}(t)\mu_{n+1}h + o(h)$$

or

$$\frac{p_n(t+h) - p_n(t)}{h} = -(\lambda_n + \mu_n)p_n(t) + \lambda_{n-1}p_{n-1}(t) + \mu_{n+1}p_{n+1}(t) + \frac{o(h)}{h}, \quad (4.3)$$

and taking limits $h \rightarrow 0$, we have

$$p_n'(t) = -(\lambda_n + \mu_n)p_n(t) + \lambda_{n-1}p_{n-1}(t) + \mu_{n+1}p_{n+1}(t), n \geq 1 \quad (4.4)$$

For $n=0$, we have

$$\begin{aligned} p_0(t+h) &= p_0(t) \{1 - \lambda_0 h + o(h)\} + p_1(t) \{1 - \lambda_0 + o(h)\} \{\mu_1 h + o(h)\} \\ &= p_0(t) - \lambda_0 h p_0(t) + \mu_1 h p_1(t) \end{aligned} \quad (4.5)$$

Or

$$\frac{p_0(t+h) - p_0(t)}{h} = -\lambda_0 p_0(t) + \mu_1 p_1(t) + \frac{o(h)}{h}$$

whence we get

$$p_0'(t) = -\lambda_0 p_0(t) + \mu_1 p_1(t) \quad (4.6)$$

If at epoch $t=0$, there were $i (\geq 0)$ individuals, then the initial condition is

$$p_n(0) = 0, \quad n \neq i; \quad p_i(0) = 1.$$

The equations (4.4) and (4.6) are the equations of the birth and death process. The birth and death processes play an important role in queueing theory. They also have interesting applications in diverse other fields such as economics, biology, ecology, reliability theory etc.

2.3.5 Birth and Death Rates

Some particular values of λ_n and μ_n are of special interest. When $\lambda_n = \lambda$, i.e. λ_n is independent of the population size n , then the increase may be thought of as due to an external source such as immigration.

When $\lambda_n = nh$, we have the case of (linear) birth; $\lambda_n h = n\lambda h$ may be considered as the probability of one birth in an interval of length h given that n individuals are present (at the instant from which the interval commences), the probability of one individual giving a birth being λh , (i.e. rate of birth in unit interval is λ per individual). Here $\lambda_0 = 0$.

When $\mu_n = \mu$, the decrease may be thought of as due to a factor such as emigration. When $\mu_n = n\mu$, we have the case of (linear) death, the rate of death in unit interval being μ per individual.

2.4 Markov Processes with Discrete State Space (Continuous Time Markov Chain)

Let $X(t)$ be a continuous parameter Markov Process with discrete state space $N = \{0, 1, 2, \dots, i, \dots, j, \dots\}$. Further let $X(t)$ be time-homogeneous then the (transition) probability of a transition from state i to state j during the time interval from epoch T to epoch $T+t$ does not depend on the initial time T but depends only on the elapsed time t and on the initial and terminal states i and j . We can thus write

$$P\{X(T+t)=j|X(T)=i\}=p_{ij}(t), \quad (5.1)$$

$$i, j=0, 1, 2, \dots, t \geq 0.$$

In particular,

$$\Pr\{X(t)=j|X(0)=i\}=p_{ij}(t).$$

We have

$$0 \leq p_{ij}(t) \leq 1 \text{ for each } i, j, t$$

and

$$\sum_j p_{ij}(t) = 1.$$

Let $p_j(t) = \Pr\{X(t)=j\}$ be the state probability at epoch t ; then

$$\begin{aligned} p_j(t) &= \Pr\{X(t)=j\} \\ &= \sum_i \Pr\{X(t)=j \text{ and } X(0)=i\} \\ &= \sum_i \Pr\{X(0)=i\} \Pr\{X(t)=j|X(0)=i\} \\ &= \sum_i \Pr\{X(0)=i\} p_{ij}(t). \end{aligned} \quad (5.2)$$

We have

$$\sum_j p_j(t) = 1 \text{ for each } t \geq 0$$

Let us denote the transition probability matrix by

$$P(t) = (p_{ij}(t)). \quad (5.3)$$

Setting $p_{ij}(0) = \delta_{ij}$, we get

$$\begin{aligned} P(0) &= I. \\ (5.4) \end{aligned}$$

We shall assume here that the functions $p_{ij}(t)$ are continuous and differentiable for $t \geq 0$.

The waiting time for a change of state:

Suppose that $X(t)$ is a homogeneous Markov process and that at time $t_0=0$ the state of the process $X(t_0)=X(0)=i$ is known. The time taken for a change of state i is a random variable, say τ , which is called the waiting time for a change of state from state i .

We have

$$\begin{aligned} \Pr\{\tau > s+t | X(0)=i\} \\ &= \Pr\{\tau > s+t | X(0)=i, \tau > s\} \Pr\{\tau > s | X(0)=i\} \\ &= \Pr\{\tau > s+t | X(s)=i\} \Pr\{\tau > s | X(0)=i\}, \end{aligned}$$

If we denote

$$\bar{F}(t) = \Pr\{\tau > t | X(0)=i\}, t > 0$$

then the above can be written as

$$\bar{F}(s+t) = \bar{F}(s) \bar{F}(t), \text{ for } s, t > 0.$$

Which is satisfied iff $\bar{F}(t)$ is the form

$$\bar{F}(t) = e^{-\lambda t}, t > 0, \lambda > 0.$$

Thus the waiting τ has exponential distribution with parameter λ , which is called the transition density from state i . The distribution is the same for all i .

2.5 Chapman- Kolmogorov Equations

The transition probability $p_{ij}(t+T)$ is the probability that given the state was i at epoch 0, it is state j at epoch $t+T$; but in passing from state i to state j in time $(t+T)$ the process passes through some state k in time t . Thus

$$\begin{aligned} P_{ij}(t+T) &= \sum_k \Pr\{X(t+T)=j, X(t)=k | X(0)=i\} \\ &= \sum_k \Pr\{X(0)=i, X(t)=k, X(t+T)=j\} \Pr\{X(0)=i\} \\ &= \sum_k \frac{\Pr\{X(0)=i, X(t)=k\}}{\Pr\{X(0)=i\}} \times \frac{\Pr\{X(0)=i, X(t)=k, X(t+T)=j\}}{\Pr\{X(0)=i, X(t)=k\}} \\ &= \sum_k \Pr\{X(t)=k | X(0)=i\} \Pr\{X(t+T)=j | X(0)=i, X(t)=k\}. \end{aligned}$$

Since $\{X(t)\}$ is a Markov process,

$$\begin{aligned}\Pr\{X(t+T)=j|X(0)=i, X(t)=k\} \\ &= \Pr\{X(t+T)=j|X(t)=k\} \\ &= p_{kj}(T).\end{aligned}$$

Thus we get the relation

$$p_{ij}(t+T) = \sum_k p_{ik}(T)p_{kj}(T), \quad (5.5)$$

Which holds for all states i, j and $t \geq 0, T \geq 0$; it is called the Chapman-Kolmogorov equation. In matrix notation it can be written as

$$P(t+T) = P(t)P(T). \quad (5.6)$$

The equation (5.5) and (5.6) respectively which hold for discrete space Markov chain.

Denote the right-hand side derivative at zero by

$$\begin{aligned}a_{ij} &= \frac{d}{dt} p_{ij}(t)|_{t=0}; i \neq j \\ a_{ii} &= \frac{d}{dt} p_{ii}(t)|_{t=0}\end{aligned} \quad (5.7)$$

Then

$$a_{ij} = \lim_{\Delta t \rightarrow 0} \frac{p_{ij}(\Delta t) - p_{ij}(0)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{p_{ij}(\Delta t)}{\Delta t}$$

or

$$p_{ij}(\Delta t) = a_{ij}\Delta t + o(\Delta t), i \neq j \quad (5.8)$$

and

$$a_{ii} = \lim_{\Delta t \rightarrow 0} \frac{p_{ii}(\Delta t) - p_{ii}(0)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{p_{ii}(\Delta t) - 1}{\Delta t}$$

or

$$p_{ii}(\Delta t) = 1 + a_{ii}(\Delta t) + o(\Delta t). \quad (5.9)$$

It can be seen from the above relations that $a_{ij} \geq 0, i \neq j$ and $a_{ii} < 0$.

From

$$\sum_j p_{ij}(t) = 1, \text{ using (5.7), we get}$$

$$\sum_j a_{ij} = 0$$

Or

$$\sum_{j \neq i} a_{ij} = -a_{ii} \quad (5.10)$$

The quantities a_{ij} are called transition densities and the matrix

$$A = (a_{ij})$$

is called the transition density matrix or rate matrix of the process. The matrix is such that

- (1) its off-diagonal elements are non-negative and the diagonal elements are negative;
- (2) the sum of the elements of each row is zero, the sum of the off-diagonal elements being equal in magnitude but opposite in sign to the diagonal elements.

Differentiating (5.5) with respects to T , we get

$$p'_{ij}(t+T) = \frac{\partial}{\partial T} p_{ij}(t+T) = \sum_k p_{ik}(t) \frac{d}{dT} p_{kj}(T).$$

Putting $T=0$, we get

$$p'_{ij}(t) = \sum_k p_{ik}(t) a_{kj} \quad (5.11)$$

Or, in matrix notation

$$P'(t) = P(t)A \quad (5.11 a)$$

Similarly we can get

$$\frac{d}{dT} p_{ij}(T) = \sum_k a_{ik} p_{kj}(T)$$

Replacing T by t, we can write this as

$$p'_{ij}(t) = \sum_k a_{ik} p_{kj}(t) \quad (5.12)$$

Or

$$P'(t) = AP(t). \quad (5.12 a)$$

Equations (5.11) and (5.12) which give Chapman-Kolmogorov equations as differential equations are called respectively Forward and Backward Kolmogorov equations.

Solution of the Equations for a Finite State Process

When the rate matrix is given, the equations (5.11) or (5.11 a) together with the initial conditions $p_{ij} = \delta_{ij}$ (or $P(0) = I$) yield as solution the unknown probabilities $p_{ij}(t)$. We consider below a method of solution for a process with finite number of states. From (5.11 a) we see at once that the solutions can be written in the form

$$P(t) = P(0)e^{At} = e^{At} \quad (5.13)$$

Where the matrix

$$e^{At} = I + \sum_{n=1}^{\infty} \frac{A^n t^n}{n!} \quad (5.13 a)$$

Assume that eigenvalues of A are all distinct. Then from the spectral theorem of matrices, we have

$$A = HDH^{-1}$$

Where H is a non-singular matrix (formed with the right eigen vectors of A) and D is the diagonal matrix having for its diagonal elements the eigenvalues of A. Now, λ_i is an eigenvalue of A and if $d_i \neq 0$, $i=1, \dots, m$ are the other distinct eigenvalues, then

$$D = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & d_1 & \cdots & 0 \\ \cdots & & \cdots & \cdots \\ \cdots & & \cdots & \cdots \\ 0 & 0 & \cdots & d_m \end{pmatrix}$$

We then have

$$D^n = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & d_1^n & \cdots & 0 \\ \cdots & & \cdots & \cdots \\ \cdots & & \cdots & \cdots \\ 0 & 0 & \cdots & d_m^n \end{pmatrix}$$

And

$$A^n = HD^n H^{-1}$$

Substituting in (5.13), we get

$$\begin{aligned} P(t) &= I + \sum_{n=1}^{\infty} \frac{(HD^n H^{-1})t^n}{n!} \\ &= H \left\{ I + \sum_{n=1}^{\infty} \frac{D^n t^n}{n!} \right\} H^{-1} \\ &= He^{Dt} H^{-1} \end{aligned} \tag{5.14}$$

Where

$$e^{Dt} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & e^{d_1 t} & \cdots & 0 \\ \cdots & & \cdots & \cdots \\ \cdots & & \cdots & \cdots \\ 0 & 0 & \cdots & e^{d_m t} \end{pmatrix}$$

The right-hand side of (5.14) gives explicit solution of the matrix P(t). Note that even in the general case when the eigenvalues of A are not necessarily distinct, a canonical representation of $A = SZS^{-1}$ exists. Using this, P(t) can be obtained in a modified form.

2.6 Poisson Process and Related Distributions

2.6.1 Inter-arrival Time

With a Poisson process, $\{N(t), t \geq 0\}$, where $N(t)$ denotes the number of occurrences of an event E by epoch t , there is associated a random variable—the interval X between two successive occurrences of E . We proceed to show that X has a negative exponential distribution.

Theorem 2.2. The interval between two successive occurrences of a Poisson process $\{N(t), t \geq 0\}$ having parameter λ has a negative exponential distribution with mean $1/\lambda$.

Proof: Let X be the random variable representing the interval between two successive occurrences of $\{N(t), t \geq 0\}$ and let $\Pr(X \leq x) = F(x)$ be its distribution function.

Let us denote two successive events by E_i and E_{i+1} and suppose that E_i occurred at instant t_i .

Then

$$\begin{aligned}
 \Pr\{X > x\} &= \Pr\{E_{i+1} \text{ did not occur in } (t_i, t_i+x) \text{ given that } E_i \\
 &\quad \text{occurred at the instant } t_i\} \\
 &= \Pr\{E_{i+1} \text{ did not occur in } (t_i, t_i+x) | N(t_i) = i\} \\
 &\quad \text{(because of the postulate of independence)} \\
 &= \Pr\{\text{no occurrence takes place in an interval } (t_i, t_i+x) \text{ of} \\
 &\quad \text{length } x | N(t_i) = i\} \\
 &= \Pr\{N(x) = 0 | N(t_i) = i\} = p_0(x) = e^{-\lambda x}, \quad x > 0. \quad (6.1)
 \end{aligned}$$

Since i is arbitrary, we have for the interval X between any two successive occurrences,

$$F(x) = \Pr(X \leq x) = 1 - \Pr\{X > x\} = 1 - e^{-\lambda x}, \quad x > 0.$$

The density function is

$$f(x) = F'(x) = \lambda e^{-\lambda x}, \quad x > 0. \quad (6.2)$$

It can be further proved that if X_i denotes the interval between E_i and E_{i+1} , $i = 1, 2, \dots$, then X_1, X_2, \dots are also independent.

Theorem 2.3 Given that only one occurrence of a Poisson process $N(t)$ has occurred by epoch T , then the distribution of the time interval γ in $[0, T]$ in which it occurred is uniform in $[0, T]$, i.e.

$$\Pr\{t < \gamma \leq t + dt \mid N(t) = 1\} = \frac{dt}{T}, 0 < t < T.$$

Proof: We have

$$\Pr\{t \leq \gamma \leq t + dt\} = \lambda e^{-\lambda t} dt.$$

$$\Pr\{N(t) = 1\} = e^{-\lambda t} (\lambda T),$$

$$\Pr\{N(T) = 1 \mid \gamma = t\} = e^{-\lambda(T-t)}.$$

the last one being the probability that there was no occurrence in the interval of length $(T-t)$. Hence

$$\begin{aligned} & \Pr\{t < \gamma < t + dt \mid N(t) = 1\} \\ &= \Pr\{t < \gamma < t + dt \text{ and } N(t) = 1\} / \Pr\{N(t) = 1\} \\ &= \Pr\{t < \gamma < t + dt\} \Pr\{N(t) = 1 \mid \gamma = t\} / \Pr\{N(t) = 1\} \\ &= \lambda e^{-\lambda t} dt e^{-\lambda(T-t)} / e^{-\lambda t} (\lambda T) = dt / T. \end{aligned}$$

It follows that $\Pr\{\gamma \leq s \mid N(T) = 1\} = \frac{s}{T}$

It may be said that a Poisson process distributes points at random over the infinite interval $[0, \infty]$ in the same way as the uniform distribution distributes points at random over a finite interval $[a, b]$.

CHAPTER 3

RETRIAL DEMAND INVENTORY SYSTEM IN SUPPLY CHAIN

3.1 Introduction

This Chapter deals with an integrated inventory control system that consists of one manufacturer (MF), one Distribution centre (DC), and one retailer. An (s, S) type inventory system with Poisson demand and exponentially distributed lead times are assumed at retailer and DC respectively. The demands occurring during the stock out period and the unsatisfied customer demand are permitted to enter into an orbit of size N . Such unsatisfied customer demand is known as Retrial demand.

A retrial demand is similar to an ordinary demand which has the following additional characteristics. When an arriving customer finds that an inventory is empty, he joins finite buffer called orbit from which he tries to access the inventory with given probabilistic or deterministic policy.

3.2 Model -I

3.2.1 Model Description

The two-echelon inventory control system in considered in this section is defined as follows:

A three level supply chain consisting of a single product, one manufacturing facility, one Distribution centre (DC) and one retailer. The demands initiated at retailer node follow Poisson process with parameter $\lambda (> 0)$.

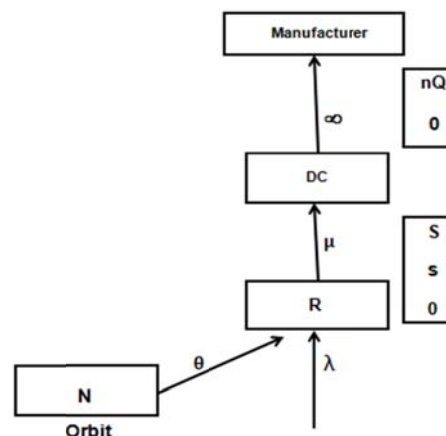


Figure 3.1: Three level inventory system in supply chain

The lead time for the replenishment of items are exponentially distributed with parameter μ (> 0). The retailer follows (s, S) policy and the distributor follow $(0, nQ)$ policy for maintaining their inventories. The unsatisfied customer at retailer node are treated as retrial customers and they are waiting in the orbit with finite capacity N . These orbiting demands compete for their demands according to an exponentially distribution with rate θ (> 0). Even though we have adopted two different policies in the Supply Chain, the distributors policy depends upon the retailers policy. The model minimizes the total cost incurred at all the locations of the system. The system performance measures and the total cost are computed in the steady state. The results are illustrated numerically.

3.2.2 Analysis

Let $X(t)$, $Y(t)$ and $Z(t)$ respectively denote the number of retrial demands in the orbit, the on hand inventory level at the retailer node and Distribution centre at time t respectively. From the assumptions on the input and output processes, clearly

$$I(t) = \{(X(t), Y(t), Z(t)) : t \geq 0\}$$

is a Markov process with state space

$$E = \{(i, j, k) / i = 0, 1, 2, \dots, N; j = S, S-1, S-2, \dots, 1, 0; k = Q, 2Q, \dots, nQ\}.$$

The infinitesimal generator of this process

$$A = (a(i, j, k : l, m, n)), (i, j, k), (l, m, n) \in E$$

can be obtained from the following arguments.

- The primary arrival of demand to the retailer node makes a transition in the Markov process from (i, j, k) to $(i, j-1, k)$ with intensity of transition λ .
- The arrival of a demand at retailer node from orbit makes a transition in the Markov Process from (i, j, k) to $(i-1, j-1, k)$ with intensity of transition $i\theta$.
- Replenishment of inventory at retailer node makes a transition from (i, j, k) to $(i, j+Q, k-Q)$ with rate of transition μ .

Here the matrices A_i , B_i and C are the square matrices of order $(S+1)n$ and note that the matrix D is the square matrix of order n .

(a) Steady State Analysis

Since the state space is finite and irreducible, hence the stationary probability vector Π for the generator A always exists and satisfies $\Pi.A = 0$ with $\Pi e = 1$

The vector Π can be represented by

$$\Pi = (\Pi^{<0>}, \Pi^{<1>}, \Pi^{<2>}, \dots, \Pi^{<N>})$$

where, $0 \leq i \leq N$

$$\Pi^{<i>} = (\pi^{<<i,0>>}, \pi^{<<i,1>>}, \dots, \pi^{<<i,S>>})$$

$$\Pi^{<<i,j>>} = (\pi^{<<<i,j,Q>>}, \pi^{<<<i,j,2Q>>}, \dots, \pi^{<<<i,j,nQ>>}), j = 0, 1, 2, \dots, S.$$

Now the structure of A shows, the model under study is a finite birth death model in the Markovian environment. Hence we use the Gaver algorithm for computing the limiting probability vector. For the sake of completeness we provide the algorithm here.

Algorithm:

1. Determine recursively the matrix D_n , $0 \leq n \leq N$ by using

$$D_0 = A_0 \tag{3.1}$$

$$D_n = A_n + B_n(-D_{n-1}^{-1})C, \quad n = 1, 2, \dots, K \tag{3.2}$$

2. Solve the system

$$\Pi^{<N>} D_N = 0. \tag{3.3}$$

3. Compute recursively the vector $\Pi^{<n>}$, $n = N-1, \dots, 0$ using

$$\Pi^{<n>} = \Pi^{<n+1>} B_{n+1}(-D_n^{-1}), n = n-1, \dots, 0. \tag{3.4}$$

4. Re-normalize the vector Π , using

$$\Pi e = 1. \tag{3.5}$$

3.2.3 Performance Measures

In this section the following system performance measures in steady for the proposed inventory system is computed.

(a) Mean Inventory Level

Let I_R denote the expected inventory level in the steady state at retailer node and I_D denote the expected inventory level at distribution centre.

$$I_R = \sum_{i=0}^N \sum_{j=1}^S \sum_{k=Q}^{nQ} j \Pi^{<<<i,j,k>>>} \quad (3.6)$$

$$I_D = \sum_{i=0}^N \sum_{j=0}^S \sum_{k=Q}^{nQ} k \Pi^{<<<i,j,k>>>} \quad (3.7)$$

(b) Expected Number of Demands in the Orbit

Let $E(O)$ denote the expected number of customer in the orbit which is given by

$$E(O) = \sum_{i=1}^N \sum_{j=0}^S \sum_{k=Q}^{nQ} i \Pi^{<<<i,j,k>>>} \quad (3.8)$$

(c) Mean Reorder Rate

The mean reorder rate at retailer node and distributor node are is given by

$$r_R = \sum_{i=0}^N \sum_{k=Q}^{nQ} (\lambda + i\theta) \Pi^{<<<i,s+1,k>>>} \quad (3.9)$$

$$r_D = \mu \sum_{i=1}^N \sum_{j=0}^s \Pi^{<<<i,j,Q>>>} \quad (3.10)$$

(d) Shortage Rate

Shortage occurs only at retailer node and the shortage rate for the retailer is denoted by α_R and which is given by

$$\alpha_R = \sum_{i=0}^N \sum_{k=Q}^{nQ} \lambda \Pi^{<<<i,0,k>>>} + \sum_{i=1}^N \sum_{k=Q}^{nQ} i\theta \Pi^{<<<i,0,k>>>} \quad (3.11)$$

3.2.4 Cost Analysis

In this section the cost structure for the proposed models by considering the minimization of the steady state total expected cost per time is analyzed.

The long run expected cost rate for the model is defined to be

$$TC(S, s, n) = h_R I_R + h_D I_D + k_R r_R + k_D r_D + g_R \alpha_R + c_0 E(O) \quad (3.12)$$

h_R - denote the inventory holding cost/ unit / unit time at retailer node

h_D - denote the inventory holding cost/ unit / unit time at distribution centre

k_R - denote the setup cost/ order at retailer node

k_D - denote the setup cost/ order at distribution node

g_R - denote the shortage cost/ unit shortage at retailer node

c_0 - denote waiting cost of a customer in the orbit / unit time.

Although the convexity of the cost function $C(s, Q)$, is not proved, our any experience with considerable number of numerical examples indicates that $C(s, Q)$ for fixed $Q (> s)$ appears to be locally convex in s . For large number of parameter, the calculation of $C(s, Q)$ revealed a convex structure.

Hence, a numerical search procedure is adopted to obtain the optimal value s for each S . Consequently, the optimal $Q (= S-s)$ and $M (= nQ)$ are obtained. A numerical example with sensitivity analysis of the optimal values by varying the different cost parameters is presented below.

3.2.5 Numerical Illustration

In this section, the problem of minimizing the steady state expected cost rate under the following cost structure is discussed. The holding cost is assumed that $h_R \geq h_D$, since the holding cost at distribution node is less than that of the retailer node as the rental charge may be illustrated through the following numerical example.

The results we obtained in the steady state case may be illustrated through the following numerical example,

$S = 20, N = 3, M = 80, \lambda = 3, \mu = 3, h_R = 1.1, h_D = 1.2, k_R = 1.5, k_d = 1.3$
 $g_R = 2.1, C_o = 2.3$

s	2	3	4	5*	6	7	8
Q	18	17	16	15*	14	13	12
$TC(s, Q)$	181.8857	166.299	142.294	138.13*	148.17	173.473	194.333

Table 3.1 Total cost function $TC(s, Q)$

For the inventory capacity S , the optimal reorder level s^* and optimal cost $TC(s, Q)$ are indicated by the symbol *. The Convexity of the cost function is given in the graph.

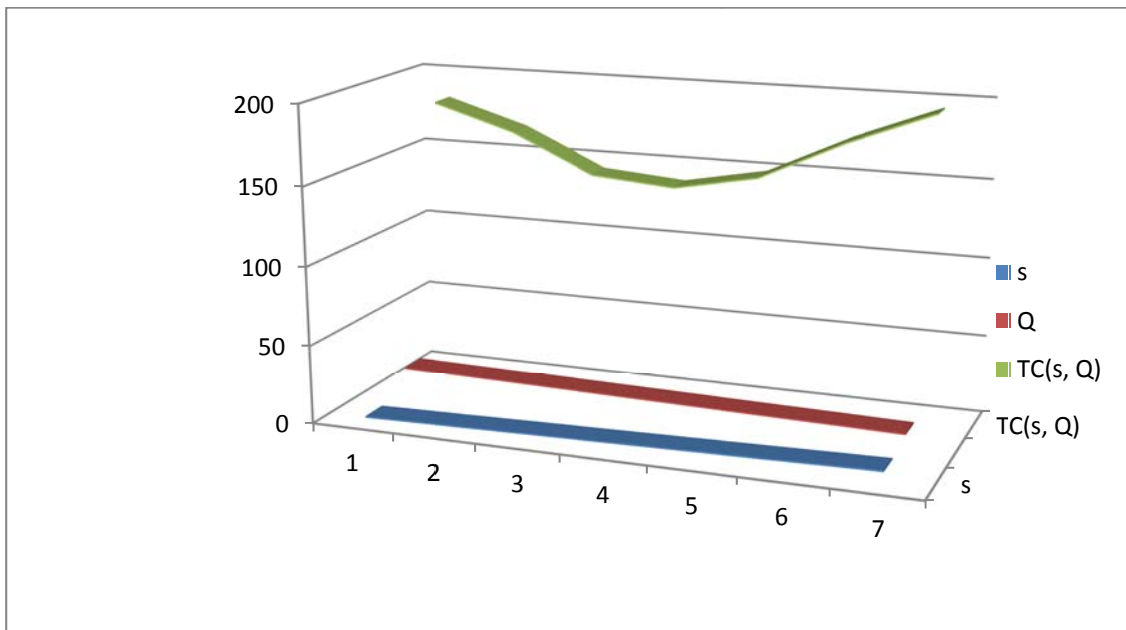


Fig. 3.1

The effect of changes in the holding costs are given in the following table.

	$h_D = 0.04$	$h_D = 0.08$	$h_D = 0.12$	$h_D = 0.16$	$h_D = 0.20$
$h_R = 0.002$	10.926279	11.476001	12.025723	12.575445	13.125166
$h_R = 0.004$	11.905344	12.455066	13.004788	13.554510	14.104231
$h_R = 0.006$	12.884409	13.434131	13.983853	14.533575	15.083296
$h_R = 0.008$	13.863474	14.413196	14.962918	15.512640	16.062362
$h_R = 0.010$	14.842539	15.392261	15.941983	16.491705	17.0414227

Table 3.2 : h_R vs h_D

The effect of changes in the setup cost and shortage cost are given in the following table.

	$g_R = 0.2$	$g_R = 0.4$	$g_R = 0.6$	$g_R = 0.8$	$g_R = 0.10$
$k_R = 10$	10.144455	10.175204	10.205953	10.236701	10.267450
$k_R = 15$	11.930169	11.960918	11.991667	12.022416	12.053165
$k_R = 20$	13.715883	13.746632	13.777381	13.808130	13.838879
$k_R = 25$	15.501598	15.532347	15.563095	15.593844	15.624593
$k_R = 30$	17.287312	17.318061	17.348810	17.379559	17.410308

Table 3.3 : k_R vs g_R

The following figures shows the effect on various parameters of the inventory system considered in this model.

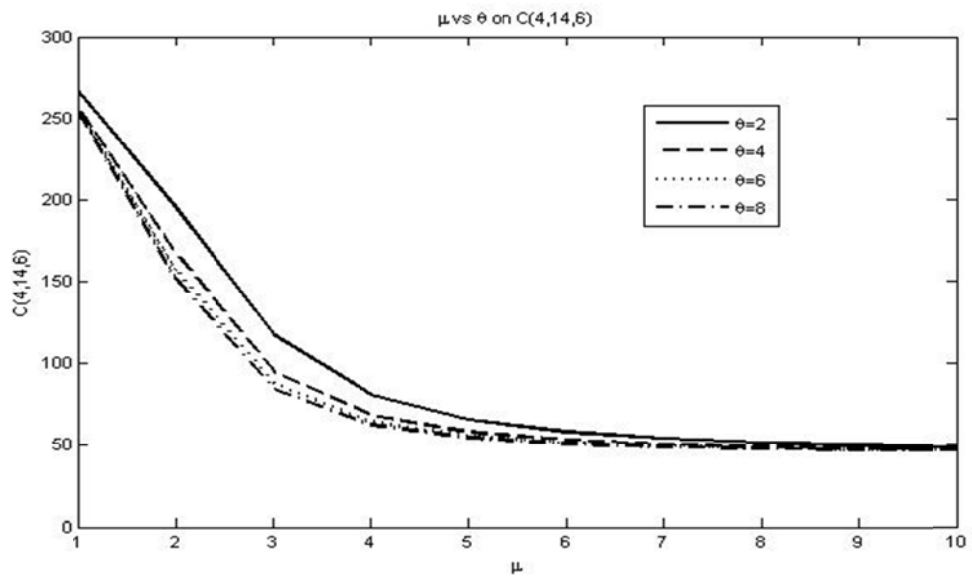


Figure 3.2 :

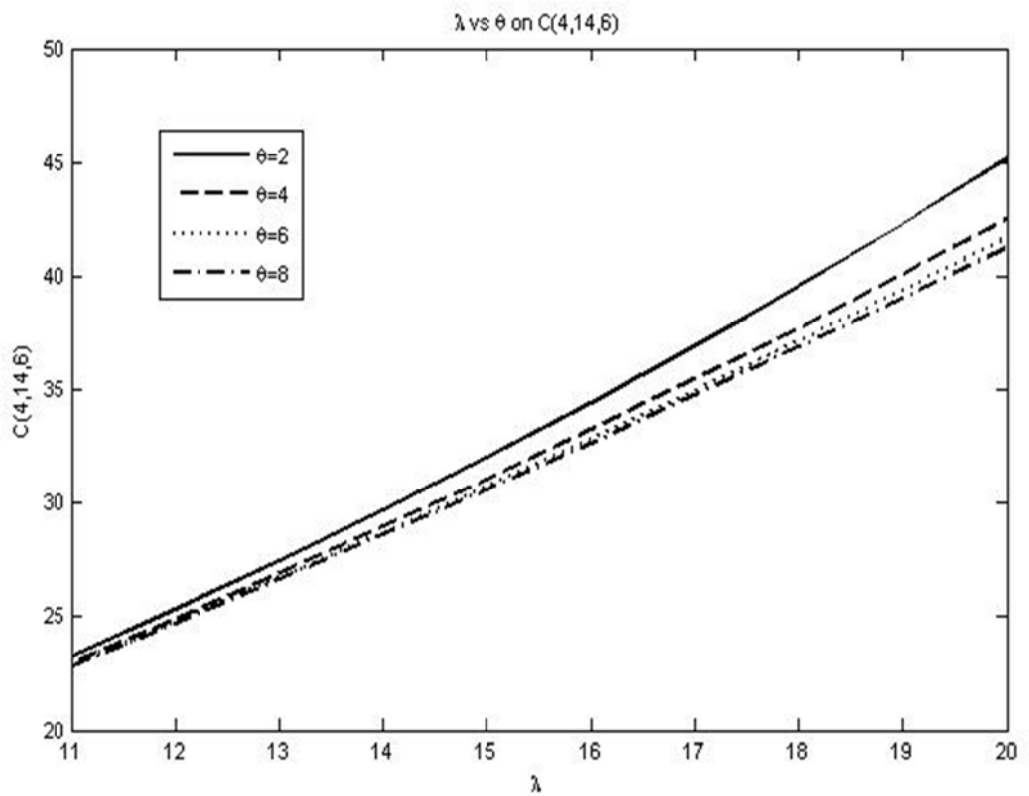


Figure 3.3:

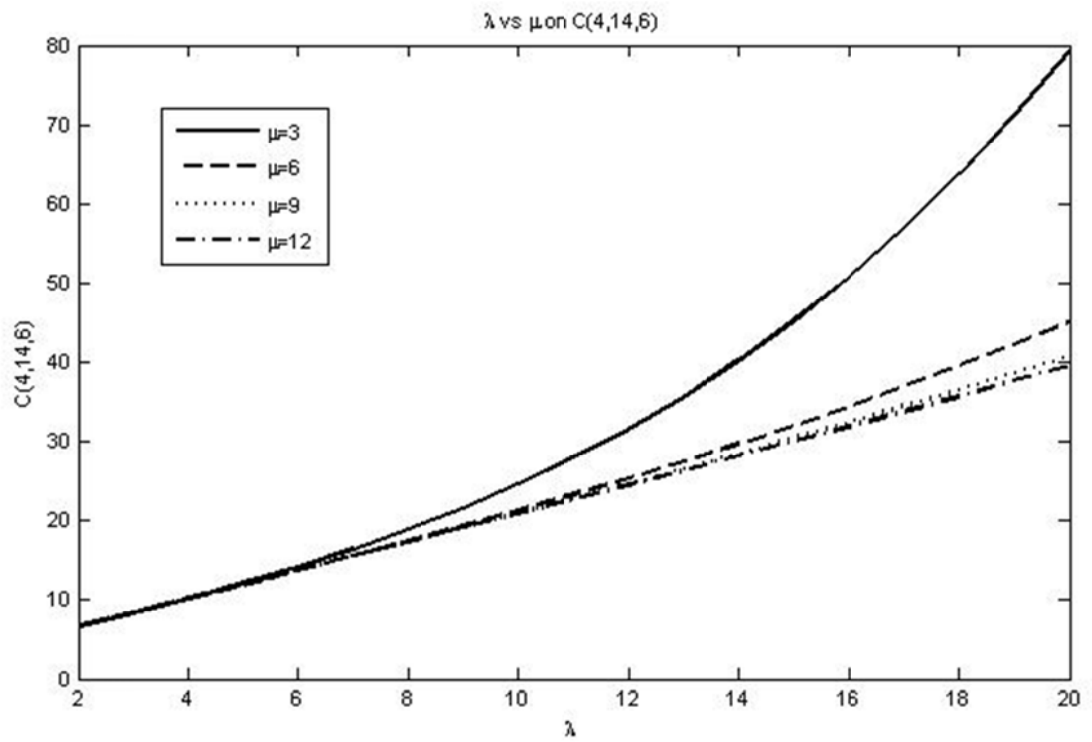


Figure 3.4 : λ vs θ on $C(4,14,6)$

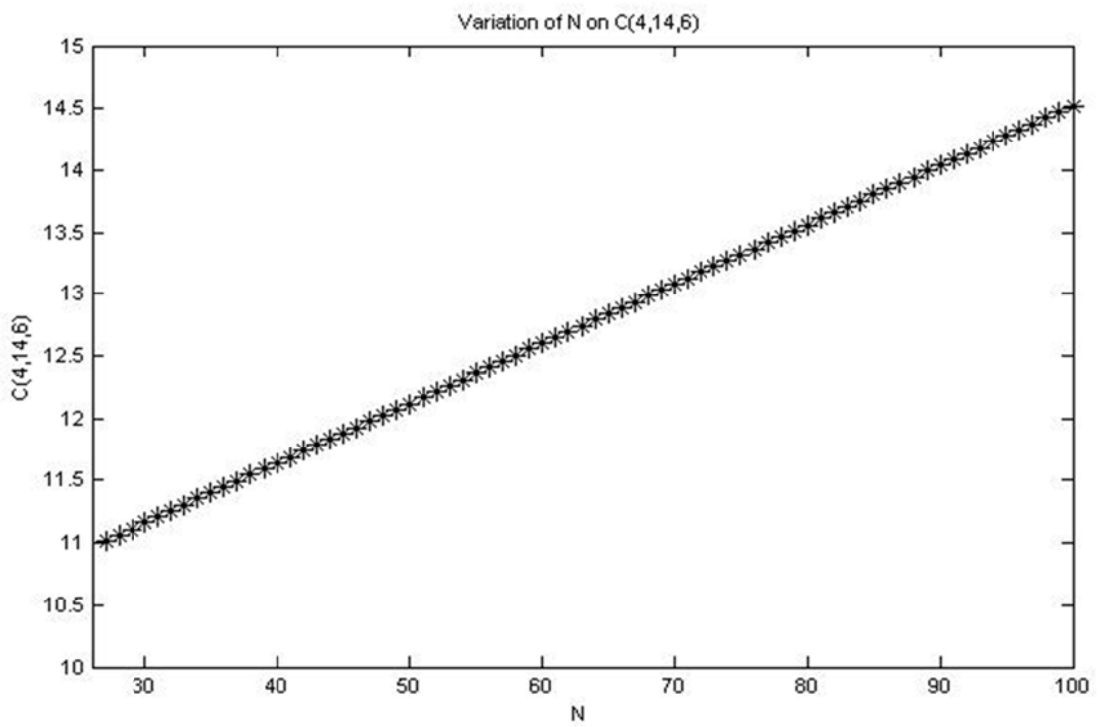


Figure 3.5 : N vs $C(4,14,6)$

From the numerical work, we conclude the following:

1. As is to be expected λ increases total cost increases, μ and θ increases total cost decreases.
2. Total cost increases when the costs h_D , h_R , g_R and k_R increases.

3.3 Model -II

3.3.1 Model Description

In this model, A three level supply chain consisting single perishable product, one manufacturing facility, one Distribution centre (DC) and one retailer. The demands initiated at retailer node follow Poisson process with parameter $\lambda (> 0)$. The replenishment of item in terms pockets from Distribution centre (DC) to retailer is exponentially distributed with parameter $\mu (> 0)$. The retailer follows (s, S) policy and the distributor follow $(0, nQ)$ policy for maintaining their inventories. It is assumed that the items are perishes only at retailer node. The life time of an item at retailer node is exponentially distributed with parameter $\gamma (> 0)$. The unsatisfied customers are treated as retrial customers and they are waiting in the orbit with finite capacity N . These orbiting demands compete for their demands according to an exponentially distribution with rate $\theta (> 0)$. Even though we have adopted two different policies in the Supply Chain, the distributors policy is depends upon the retailers policy. The model minimizes the total cost incurred at all the locations subject to the service level constraints. The system performance measures and the total cost are computed in the steady state. The results are illustrated numerically.

3.3.2 Analysis

Let $X(t)$, $Y(t)$ and $Z(t)$ respectively denote the number of retrial demands in the orbit, the on hand inventory level in the retailer node and the number of pockets in the Distribution centre at time t . From the assumptions on the input and output processes, clearly

$$I(t) = \{(X(t), Y(t), Z(t)) : t > 0\}$$

is a Markov process with state space

$$E = \{(i, j, k) / i = 0, 1, 2, \dots, N; j = S, S-1, S-2, \dots, 1, 0; k = Q, 2Q, \dots, nQ\}.$$

The infinitesimal generator of this process

$$A = (a(i, j, k : l, m, n)), \quad (i, j, k, l, m, n) \in E$$

can be obtained from the following arguments.

- The primary arrival of demand to the retailer node makes a transition in the Markov process from (i, j, k) to $(i, j-1, k)$ with intensity of transition λ .
- The arrival of a demand at retailer node from orbit makes a transition in the Markov Process from (i, j, k) to $(i-1, j-1, k)$ with intensity of transition $i\theta$.
- The item expires makes a transition from (i, j, k) to $(i, j-1, k)$ with intensity of transition $j\gamma$.
- Replenishment of inventory at retailer node makes a transition from (i, j, k) to $(i, j+Q, k-Q)$ with rate of transition μ .

Then, the infinitesimal generator has the following finite QBD structure:

$$A = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & \dots & N-2 & N-1 & N \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ \vdots \\ N-2 \\ N-1 \\ N \end{matrix} & \left(\begin{array}{ccccccc} A_0 & C & & & & & \\ B_1 & A_1 & C & & & & \\ & B_2 & A_2 & & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & A_{N-2} & C & \\ & & & & B_{N-1} & A_{N-1} & C \\ & & & & & B_N & A_N \end{array} \right) \end{matrix}$$

Where,

$$[A_i]_{k \times l} = \begin{cases} -(\lambda + \mu)I_n & l = k & k = 0, \\ -(\lambda + \mu + i\theta + k\gamma)I_n & l = k & k = 1, 2, \dots, s, \\ -(\lambda + i\theta + k\gamma)I_n & l = k & k = s+1, s+2, \dots, S, \\ (\lambda + k\gamma)I_n & l = k-1 & k = 1, 2, \dots, S, \\ D & l = k+Q & k = 0, 1, \dots, s, \\ 0 & \text{otherwise} & \end{cases}$$

$$[B_i]_{k \times l} = \begin{cases} i\theta I_n & l = k - 1 \quad k = 1, 2, \dots, S, \\ 0 & \text{otherwise} \end{cases}$$

$$[C]_{k \times l} = \begin{cases} \lambda I_n & l = k \quad k = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$D = \begin{matrix} & Q & 2Q & 3Q & \dots & nQ \\ \begin{matrix} Q \\ 2Q \\ 3Q \\ \vdots \\ nQ \end{matrix} & \left(\begin{array}{cccccc} & & & & & \mu \\ \mu & & & & & \\ & \mu & & & & \\ & & \ddots & & & \\ & & & \mu & & \end{array} \right) \end{matrix}$$

Here the matrices A_i , B_i and C are the square matrices of order $((S+1)n)$ and note that the matrix D is the square matrix of order n

(a) Steady State Analysis

Since the state space is finite and irreducible, hence the stationary probability vector Π for the generator A always exists and satisfies $\Pi A = 0$ $\Pi e = 1$

The vector Π can be represented by

$$\begin{aligned} \Pi &= (\Pi^{<0>}, \Pi^{<1>}, \Pi^{<2>}, \dots, \Pi^{<N>}) \quad \text{where } 0 \leq i \leq N \\ \Pi^{<i>} &= (\pi^{<<i,0>>}, \pi^{<<i,1>>}, \dots, \pi^{<<i,S>>}) \\ \Pi^{<<i,j>>} &= (\pi^{<<<i,j,Q>>>}, \pi^{<<<i,j,2Q>>>}, \dots, \pi^{<<<i,j,nQ>>>}), j = 0, 1, 2, \dots, S \end{aligned}$$

Now the structure of A shows, the model under study is a finite birth death model in the Markovian environment. Hence we use the Gaver algorithm for computing the limiting probability vector.

3.3.3 Performance Measures

In this section the following system performance measures in steady for the proposed inventory system is computed.

(a) Mean Inventory Level

Let I_R denote the expected inventory level in the steady state at retailer node and I_D denote the expected inventory level at distribution centre.

$$I_R = \sum_{i=0}^N \sum_{j=1}^S \sum_{k=Q}^{nQ} j \Pi^{<<<i,j,k>>>} \quad (3.13)$$

$$I_D = \sum_{i=0}^N \sum_{j=0}^S \sum_{k=Q}^{nQ} k \Pi^{<<<i,j,k>>>} \quad (3.14)$$

(b) Expected number of Demands in the orbit

Let $E(O)$ denote the expected number of customer in the orbit which is given by

$$E(O) = \sum_{i=1}^N \sum_{j=0}^S \sum_{k=Q}^{nQ} i \Pi^{<<<i,j,k>>>} \quad (3.15)$$

(c) Mean Reorder rate

The mean reorder rate at retailer node is given by

$$r_R = \sum_{i=0}^N \sum_{k=Q}^{nQ} (\lambda + i\theta + (s+1)\gamma) \Pi^{<<<i,s+1,k>>>} \quad (3.16)$$

$$r_D = \mu \sum_{i=1}^N \sum_{j=0}^s \Pi^{<<<i,j,Q>>>} \quad (3.17)$$

(d) Shortage rate

Shortage occurs only at retailer node and the shortage rate for the retailer is denoted by α_R and which is given by

$$\alpha_R = \sum_{i=0}^N \sum_{k=Q}^{nQ} \lambda \Pi^{<<<i,0,k>>>} + \sum_{i=1}^N \sum_{k=Q}^{nQ} i \theta \Pi^{<<<i,0,k>>>} \quad (3.18)$$

3.3.4 Cost Analysis

In this section the cost structure for the proposed models by considering the minimization of the steady state total expected cost per time is analyzed.

The long run expected cost rate for the model is defined to be

$$TC(S, s, n) = h_R I_R + h_D I_D + k_R r_R + g_R \alpha_R + c_0 E(o) \quad (3.19)$$

h_R - denote the inventory holding cost/ unit / unit time at retailer node

h_D - denote the inventory holding cost/ unit / unit time at distribution centre

k_R - denote the setup cost/ order at retailer node

g_R - denote the shortage cost/ unit shortage at retailer node

c_0 - denote waiting cost of a customer in the orbit / unit time.

Although the convexity of the cost function $C(s, Q)$, is not proved, our experience with considerable number of numerical examples indicates that $C(s, Q)$ for fixed $Q (> s)$ appears to be locally convex in s . For large number of parameter, the calculation of $C(s, Q)$ revealed a convex structure.

Hence, a numerical search procedure is adopted to obtain the optimal value s for each S . Consequently, the optimal $Q (= S-s)$ and $M (= nQ)$ are obtained. A numerical example with sensitivity analysis of the optimal values by varying the different cost parameters is presented below.

3.3.5 Numerical Illustration

In this section, the problem of minimizing the steady state expected cost rate under the following cost structure is discussed. The holding cost at retailer (h_R) and Distribution centre (DC) (h_D) is assumed that $h_R \geq h_D$, since the holding cost at distribution node is less than that of the retailer node as the rental charge may be illustrated through the following numerical example.

The results we obtained in the steady state case may be illustrated through the following numerical example,

$$S=20, N= 3, M = 80, \lambda = 3, \mu = 3, h_R = 1.1, h_D = 1.2, k_R = 1.5, k_d = 1.3$$

$$g_R = 2.1, C_o = 2.3, \gamma = 2.$$

s	2	3	4	5*	6	7	8
Q	18	17	16	15*	14	13	12
$TC(s, Q)$	218.2628	199.559	170.753	165.756	177.804	208.168	233.2

Table 3.5 Total cost function $TC(s, Q)$

For the inventory capacity S , the optimal reorder level s^* and optimal cost $TC(s, Q)$ are indicated by the symbol *. The Convexity of the cost function is given in the graph.

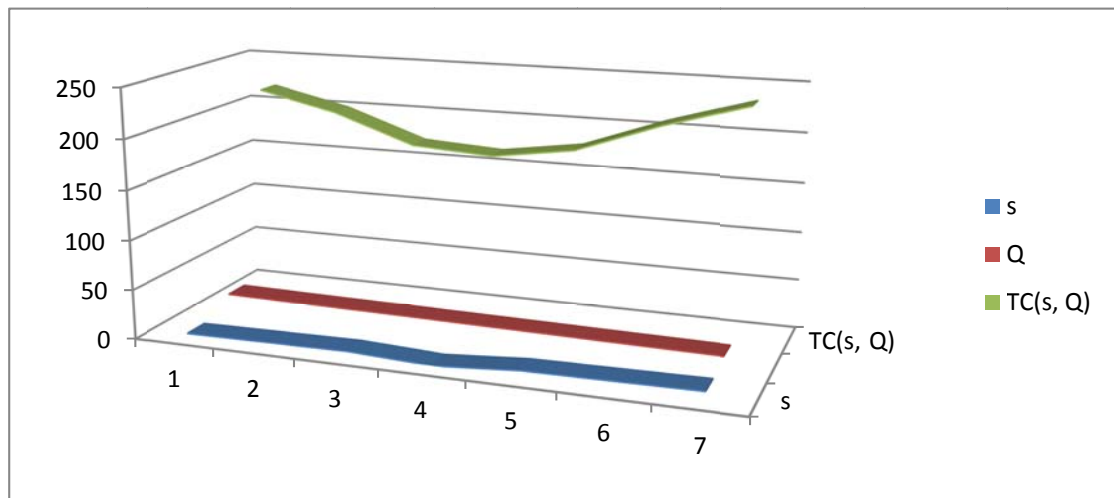


Fig. 3.6

The above figures shows the effect on various parameters of the perishable inventory system considered in this model.

	$h_D = 0.04$	$h_D = 0.08$	$h_D = 0.12$	$h_D = 0.16$	$h_D = 0.20$
$h_R =$	220.112907	220.395536	220.678164	220.960793	221.243421
$h_R =$	220.117181	220.399809	220.682438	220.965066	221.247694
$h_R =$	220.121454	220.404082	220.686711	220.969339	221.251968
$h_R =$	220.125727	220.408356	220.690984	220.973613	221.256241
$h_R =$	220.130001	220.412629	220.695258	220.977886	221.260514

Table 3.6 : h_R vs h_D

	$g_R = 0.2$	$g_R = 0.4$	$g_R = 0.6$	$g_R = 0.8$	$g_R = 0.10$
$k_D = 10$	193.290714	199.355262	205.419811	211.484359	217.548907
$k_D = 15$	204.361368	210.425917	216.490465	222.555013	228.619561
$k_D = 20$	215.432023	221.496571	227.561119	233.625667	239.690215
$k_D = 25$	226.502677	232.567225	238.631773	244.696321	250.760869
$k_D = 30$	237.573331	243.637879	249.702427	255.766975	261.831523

Table 3.7 : k_D vs g_R

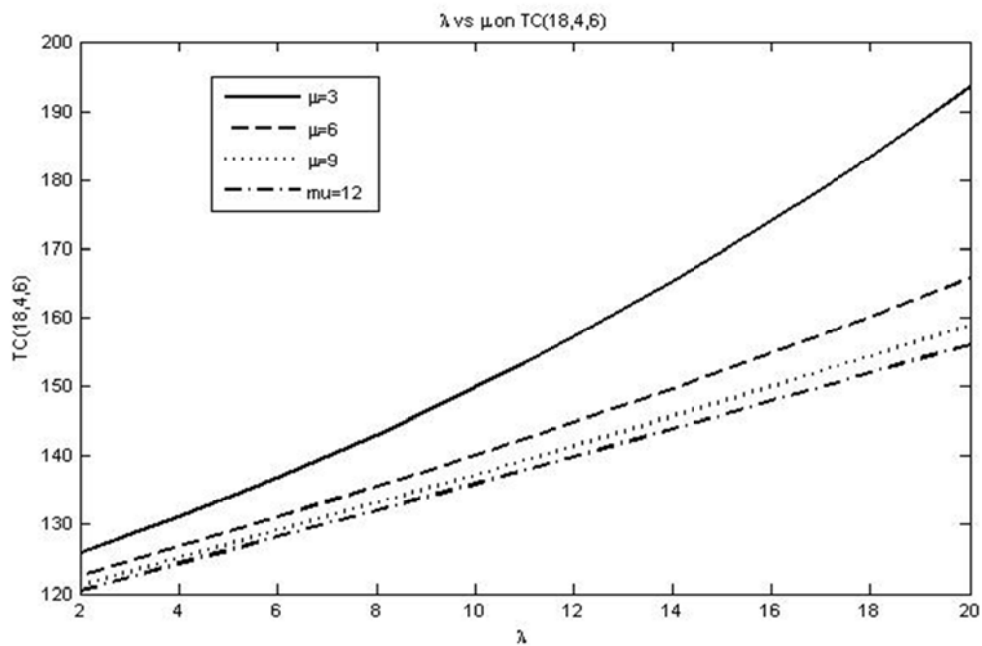


Figure 3.7: λ vs μ

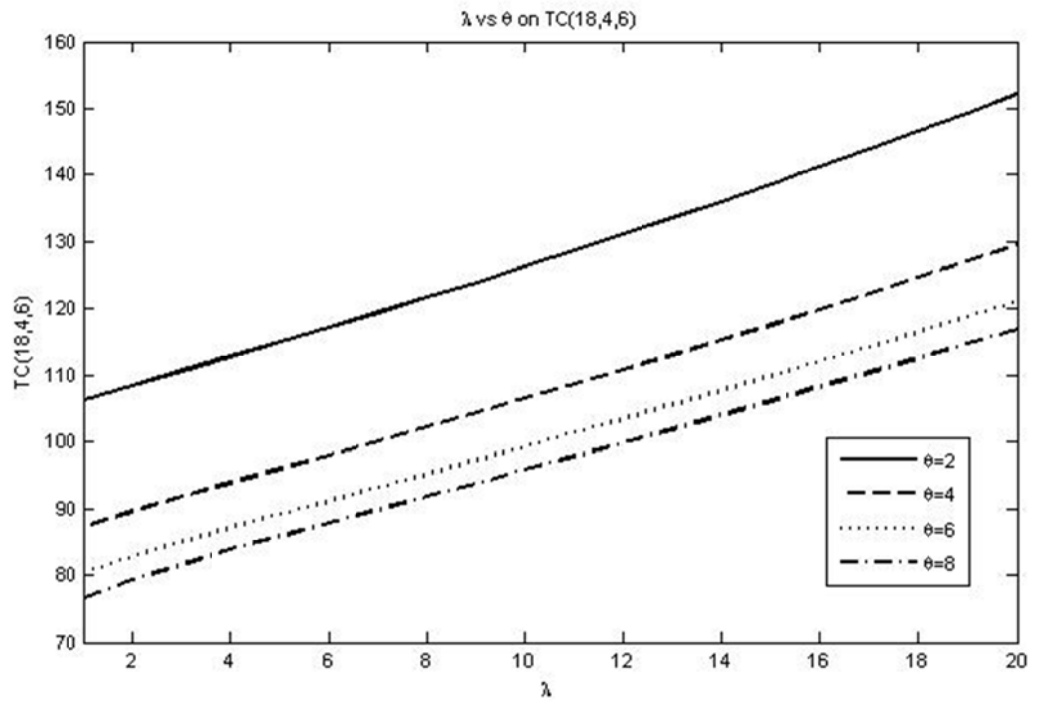


Figure 3.8 : λ vs θ

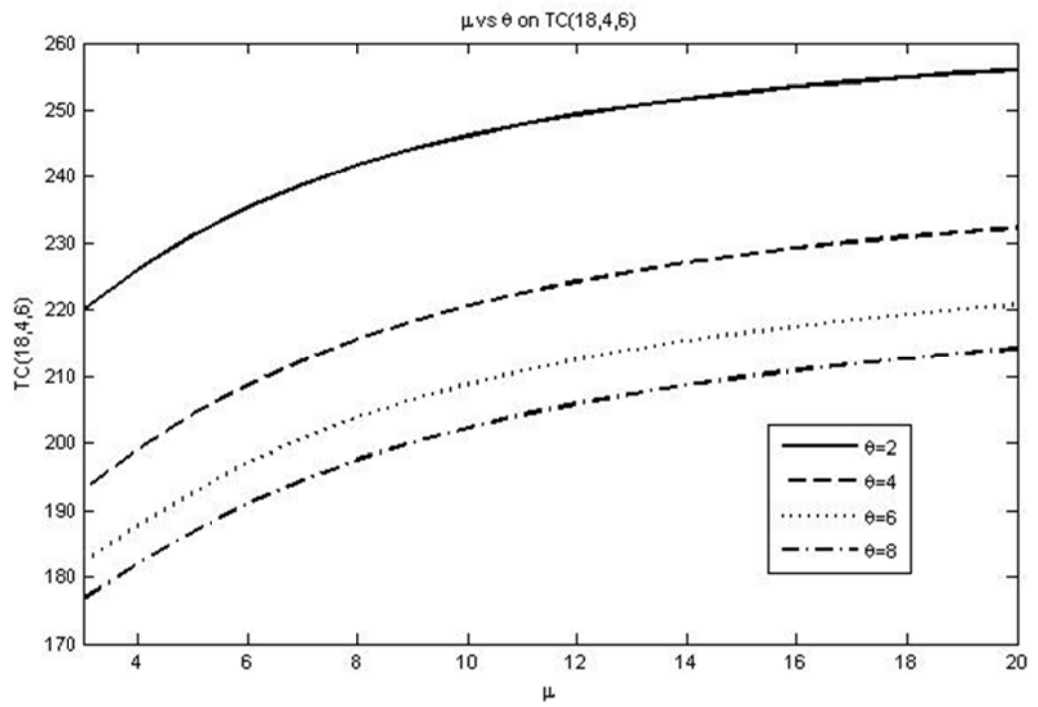


Figure 3.9 : μ vs θ

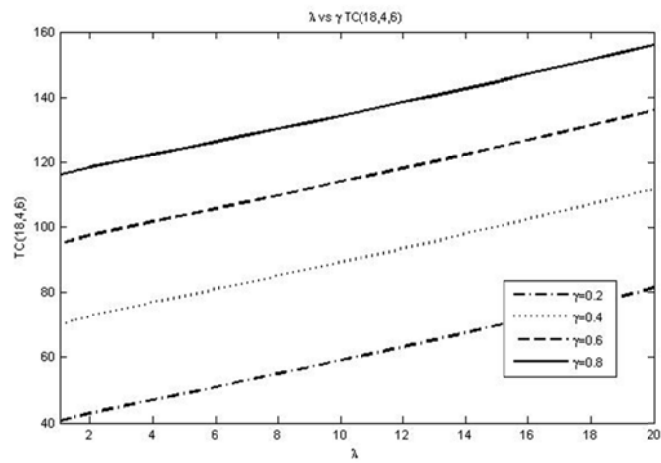


Figure 3.10 : λ vs γ

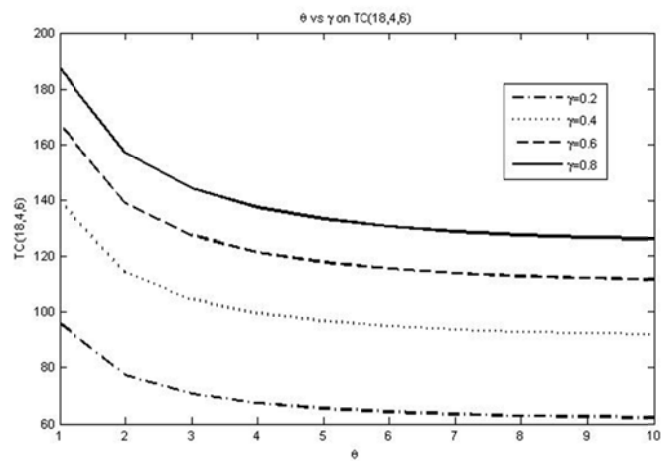


Figure 3.11 : θ vs γ on

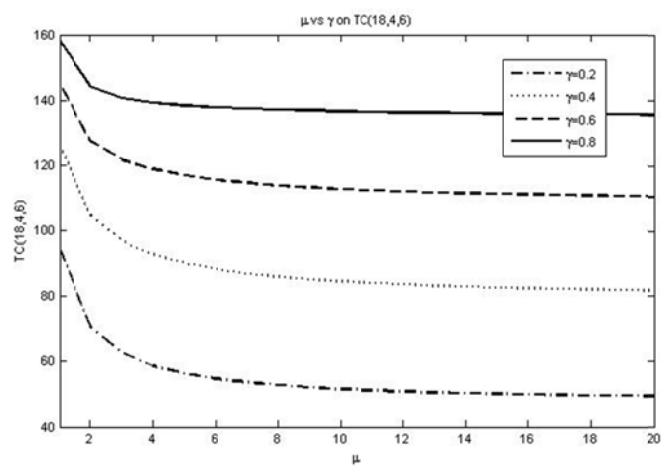


Figure 3.11 : μ vs γ on

Example: We analyzed the following in the numerical section.

1. Table 3.6 and Table 3.7 give the effect of total cost function by varying the set-up cost, holding cost, shortage cost and the back ordering cost.
2. Figure 3.6 shows that the effect of the demand rate λ and lead time μ on long run expected cost.
3. Figure 3.7 shows that the effect of the primary demand rate λ and orbiting demand rate θ on long run expected cost.
4. Figure 3.8 shows that the effect of the lead time μ and orbiting demand rate θ on long run expected cost
5. Figure 3.9 shows that the effect of the demand rate λ and perishable demand rate γ on long run expected cost.
6. Figure 3.10 shows that the effect of the orbiting demand rate θ and perishable demand rate γ on long run expected cost.
7. Figure 3.11 shows that the effect of the lead time μ and perishable demand rate γ on long run expected cost.

From the numerical work, we conclude the following:

1. As is to be expected λ increases total cost increases, μ and θ increases total cost decreases.
2. Total cost increases when the costs h_D , h_R , g_R and k_R increases.

CHAPTER 4

STOCHASTIC ANALYSIS OF PERISHABLE INVENTORY SYSTEM IN TWO-ECHELON

4.1 Introduction

In this chapter we consider a continuous review inventory control system to Multi-echelon system handling a single perishable product. Perishable inventory control in supply chain is tedious especially for food and medicine products. There for it is necessary to analyze the perishable inventory system in supply chain. Most of the researchers are working with deterministic inventory models. Few of the researchers are working under stochastic single location inventory system. This chapter analyses the perishable inventory system in two-echelon which is a building block for a supply chain system.

4.2 The Model description

The inventory control system in supply chain considered in this chapter is defined as follows.

We consider a supply chain system consisting of a manufacturer, warehousing facility, single distribution centre and n identical retailers dealing with a single finished product. These finished products moves from the manufacturer through the network consist of WH, DC, Retailers then the final customer.

A finished product is supplied from MF to WH which adopts $(0, M)$ replenishment policy then the product is supplied to DC who adopts (s, Q) policy. The demand at retailers node follows a independent Poisson distribution with rate λ_i ($i = 1, 2, 3, \dots, n$). Scanners collect sales data (demand or perish of an item) at retailer nodes and Electronic Data Interchange (EDI) allows these data to be shared to DC.

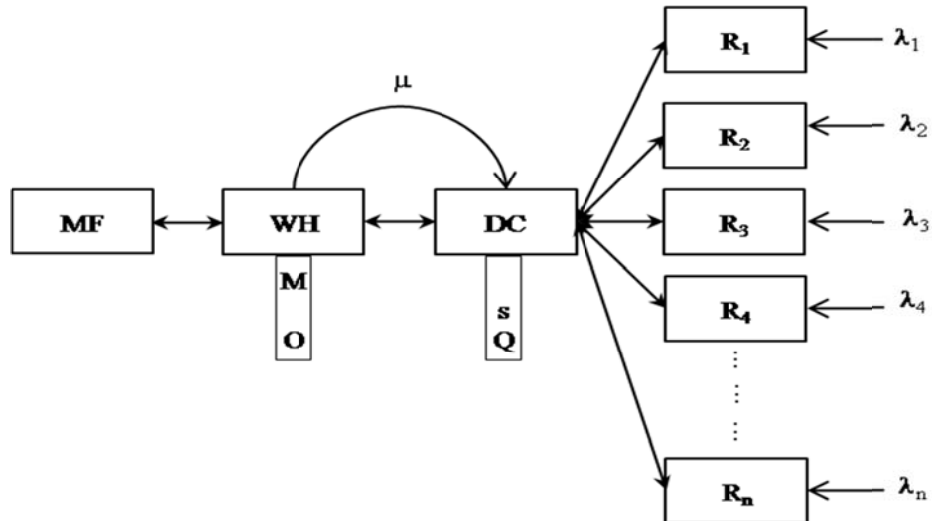


Figure 4.1 - Multi-echelon Inventory System

With the strong communication network and transport facility a unit of item is transferred to the corresponding retailer with negligible lead time. That is all the inventory transactions are managed by DC. Supply to the Manufacturer in packets of Q items is administrated with exponential lead time having parameter μ (>0). The replenishment of items in terms of pockets is made from Manufacturer to WH is instantaneous. Demands occurring during the stock out periods are assumed to be lost. The maximum inventory level at DC node S is fixed and the reorder point is s and the ordering quantity is $Q(=S-s)$ items. The maximum inventory level at Manufacturer is M ($M=nQ$).

The life time of each item supplied is exponentially distributed with parameter γ (> 0). It is assumed that the item in stock perishes only at the time of handling them at a retailer node; Decay during lead time is neglected because of tamper free packages. Decay is not considered at warehouse and distribution centre inventory.

According to the assumptions the on hand inventory levels at both nodes follows a random process.

We fix the following notations for the forthcoming analysis part of our chapter.

$[\mathbf{R}]_{ij}$: The element /sub matrix at $(i,j)^{\text{th}}$ position of \mathbf{R} .

$\mathbf{0}$: Zero matrix.

\mathbf{I} : Identity matrix.

- \mathbf{e} : A column vector of 1's of appropriate dimension.
- $\mathbf{I}_i(\mathbf{t})$: On hand inventory level at time t at location i ($i=0,1$).
- γ_i : The Perishable rate at retailer node i ($i=0,1$).
- \mathbf{k}_D : Fixed ordering cost, regardless of order size at DC node.
- \mathbf{k}_R : $\mathbf{k}_R = \bar{k}_i$, where $i=1,2,3 \dots n$. the average ordering cost at retailer nodes.
- \mathbf{k}_0 : $\mathbf{k}_D + \mathbf{k}_R$, ordering cost for integrated DC system.
- \mathbf{k}_1 : Fixed ordering cost for WH.
- \mathbf{h}_D : The holding cost per unit of item per unit time at DC
- \mathbf{h}_R : $\mathbf{h}_R = \bar{h}_i$, where $i=1, 2, 3 \dots n$. the average holding cost per unit of item per unit time at retailer nodes.
- \mathbf{h}_0 : $\mathbf{h}_D + \mathbf{h}_R$ the holding cost for integrated DC system.
- \mathbf{h}_1 : The holding cost per unit of item per unit time at WH.
- \mathbf{g}_D : The unit shortage cost at DC.
- \mathbf{g}_R : $\mathbf{g}_R = \bar{g}_i$, where $i=1, 2, 3 \dots n$. The average shortage cost per unit shortage at retailer nodes.
- \mathbf{g} : $\mathbf{g} = \mathbf{g}_D + \mathbf{g}_R$ The unit shortage cost for integrated DC system.
- $$\sum_{i=1}^k a_i = a_1 + a_2 + \dots + a_k. \quad \sum_{i=0}^{nQ} i = 0 + Q + 2Q + \dots + nQ.$$

4.3 Analysis

Let $I_0(t)$ and $I_1(t)$ denote the on-hand inventory levels at Distribution Centre and warehouse respectively at time t^+ . From the assumptions on the input and output processes, $\mathbf{I}(t) = \{(I_0(t), I_1(t) : t \geq 0\}$ is a Markov process with state space $E = \{(j, q) / j = S, (S-1), \dots, s, (s-1), \dots, 2, 1, 0, \dots, \text{ and } q = nQ, (n-1)Q, \dots, Q\}$. Since E is finite and all its states are recurrent non-null, $\{\mathbf{I}(t) : t \geq 0\}$ is an irreducible Markov process with state space E and it is an ergodic process. Hence the limiting distribution exists and is independent of the initial state.

The infinitesimal generator of this process $\mathbf{R} = (a(j, q; k, r))_{(j,q),(k,r) \in E}$ can be obtained from the following arguments.

- The arrival of a demand (or perish) for an item at Distribution centre makes a state transition in the Markov process from (j, q) to $(j-1, q)$ with intensity of transition $\lambda + j\gamma$ (where $\lambda = \lambda_i, i = 1, 2, 3, \dots, n$.)
- Replenishment of inventory at Distribution centre makes a state transition from (j, nQ) to $(j + Q, (n-1)Q)$ with rate of transition $\mu (> 0)$.

The infinitesimal generator R is given by

$$R = \begin{pmatrix} A & B & 0 & \dots & 0 & 0 \\ 0 & A & B & \dots & 0 & 0 \\ 0 & 0 & A & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & A & B \\ B & 0 & 0 & \dots & 0 & A \end{pmatrix}$$

The entries of the block partition matrix R can be written as

$$[R]_{q \times r} = \begin{cases} A & \text{if } q=r, \quad r = nQ, (n-1)Q, (n-2)Q, \dots, Q. \\ B & \text{if } q=r+Q \quad r = (n-1)Q, (n-2)Q, \dots, 2Q. \\ B & \text{if } q=nQ \\ 0 & \text{otherwise} \end{cases}$$

The sub matrices A and B are given by

$$[A]_{j \times q} = \begin{cases} \lambda + j\gamma & \text{if } q=j-1, \quad j=S, S-1, S-2, \dots, 1 \\ -(\lambda + j\gamma) & \text{if } q=j \quad j=S, S-1, S-2, \dots, (s+1) \\ -(\lambda + j\gamma + \mu) & \text{if } q=j \quad j=s, s-1, s-2, \dots, 1 \\ -\mu & \text{if } q=j \quad j=0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$[B]_{j \times q} = \begin{cases} \mu & \text{if } q=j+Q \quad j=s, s-1, s-2, \dots, 0. \\ 0 & \text{otherwise} \end{cases}$$

4.3.1 Transient analysis

Define the transition probability function $p_{j,q}(k, r; t) = \Pr\{(I_0(t), I_1(t) = (k, r) | (I_0(0), I_1(0)) = (j, q)\}$. The corresponding transient matrix function is given by $P(t) = (p_{j,q}(k, r; t))_{(j,q),(k,r) \in E}$ vector which satisfies the Kolmogorov-forward equation $P'(t) = P(t)R$, where R is the infinitesimal generator of the process $\{I(t), t \geq 0\}$. The above equation, together with initial condition $P(0) = I$, the solution can be expressed in the form $P(t) = P(0)e^{Rt} = e^{Rt}$, where the matrix expansion in power series form is $e^{Rt} = I + \sum_{n=1}^{\infty} \frac{R^n t^n}{n!}$.

4.3.2 Steady state analysis:

The structure of the infinitesimal matrix R reveals that the state space E of the Markov process $\{I(t), t \geq 0\}$, is finite and irreducible. Let the limiting probability distribution of the inventory level process be

$v_j^q = \lim_{t \rightarrow \infty} \Pr\{(I_0(t), I_1(t) = (j, q))\}_{(j,q) \in E}$, where v_j^q is the steady state probability that the system be in state (j, q) .

Let $v = (v^{nQ}, v^{(n-1)Q}, v^{(n-2)Q}, \dots, v^Q, v^0)$ denote the steady state probability distribution where $v^q = (v_S^q, v_{S-1}^q, \dots, v_S^q, v_{S-1}^q, \dots, v_0^q)$ for the system under consideration. For each (j, q) , v_j^q can be obtained by solving the matrix equation $vA = 0$ together with normalizing condition $\sum_{(j,q) \in E} v_j^q = 1$

Assuming $v_0^Q = a$, we obtain the steady state probabilities of the system states as

$${}^i Q v = (-1)^k a (BA)^k, \quad i = 1, 2, \dots, n; \quad k = n-i+1. \quad \text{where } a = e \left[\sum_{i=0}^{n-1} (-1)^i (BA^{-1})^i \right]^{-1}.$$

4.4 Operating characteristics

In this section, we derive some important system performance measures for the system under steady state.

(a) Mean Reorder Rates:

The Mean reorder rate at Distribution centre (β_0) and Manufacturer (β_1) are given by

$$\beta_0 = (\lambda + (s+1)\gamma) \sum_{q=Q}^{nQ} * v_{s+1}^q \quad \text{and} \quad \beta_1 = \mu \sum_{j=0}^s v_j^Q \quad (4.1)$$

(b) Mean Inventory Levels:

The mean inventory level in the steady state at Distribution centre (I_0) and Manufacturer (I_1) are given by

$$I_0 = \sum_{q=Q}^{nQ} * \left(\sum_{j=0}^s j \cdot v_j^q \right) \quad \text{and} \quad I_1 = \sum_{j=0}^s \left(\sum_{q=Q}^{nQ} * q \cdot v_j^q \right). \quad (4.2)$$

(c) Mean Shortage Rate:

The Shortages occurs only at retailer node. The shortage rate (α_0) is given by

$$\alpha_0 = \lambda \sum_{q=Q}^{nQ} * v_0^q \quad (4.3)$$

4.5 Algorithm

In this Section we design the algorithm to compute the long run expected inventory cost (for all the echelons).

- Step -1 : Determine the matrices A and B.
- Step-2 : Generate the R Matrix.
- Step-3 : Solve the system $\Pi R = 0$ with normalizing condition $\Sigma \Pi = 1$.
- Step-4 : Compute the operating characteristics.
- Step-5 : Compute the long run expected inventory cost.

4.6 Cost analysis

In this section we impose a cost structure for the proposed model and analyze it by the criteria of minimization of long run total expected cost per unit time.

The long run expected cost rate $C(s, Q)$ is given by

$$C(s, Q) = h_0 I_0 + h_1 I_1 + k_0 \beta_0 + k_1 \beta_1 + g \alpha_0 \tag{4.4}$$

Substituting (1),(2) and (3) in (4) we get

$$C(s, Q) = h_0 \left(\sum_{q=Q}^{nQ} * \left(\sum_{j=0}^S j \cdot v_j^q \right) \right) + h_1 \left(\sum_{j=0}^S \left(\sum_{q=Q}^{nQ} * q \cdot v_j^q \right) \right) + k_0 \left((\lambda + (s + 1) \gamma) \sum_{q=Q}^{nQ} * v_{s+1}^q \right) + k_1 \left(\mu_0 \sum_{j=0}^S v_j^Q \right) + g \left(\lambda \sum_{q=Q}^{nQ} * v_0^q \right)$$

4.7. Numerical Illustration

In the section the problem of minimizing the long run total expected cost per unit time under the following cost structure is considered for discussion. The optimum values of the system parameters s is obtained and the sensitive analysis is also done for the system.

The results we obtained in the steady state case may be illustrated through the following numerical example,

$$S = 20, M = 80, \lambda = 3, \mu = 3, h_R = 1.1, h_D = 1.2, k_R = 1.5, k_d = 1.3, g_R = 2.1, C_o = 2.3$$

The cost for different reorder level are given by

s	2	3	4	5*	6	7	8
Q	18	17	16	15	14	13	12
$TC(s, Q)$	124.013	113.385	97.0185	94.1795*	101.025	118.277	132.5

Table:4.1 Total expected cost rate as a function s and Q

For the inventory capacity S , the optimal reorder level s^* and optimal cost $TC(s, Q)$ are indicated by the symbol *. The Convexity of the cost function is given in the graph.

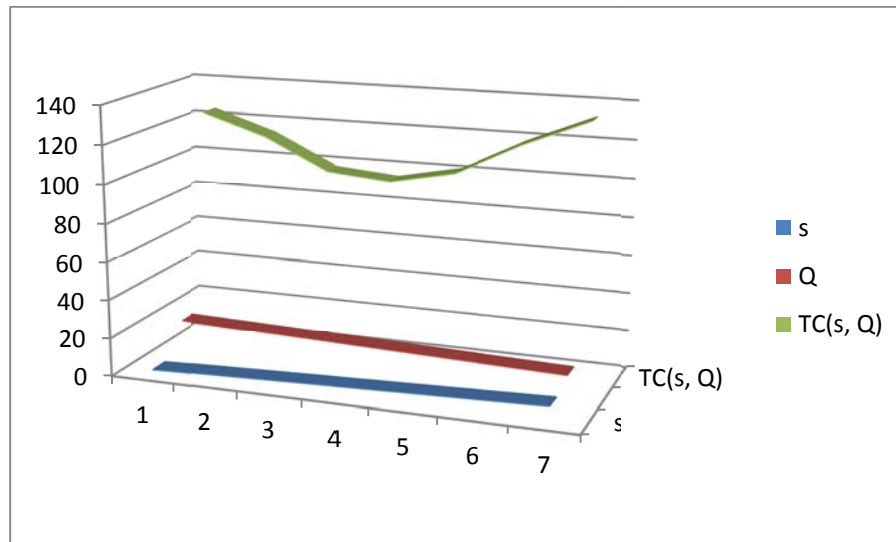


Fig 4.1:Total expected cost rate as a function $TC(s,Q)$, s and Q

4.7.1 Sensitivity Analysis

	$s=2$	$s=4$	$s=6$	$s=8$	$s=10$
S=45	111.998164	118.761642	124.799813	130.664478	136.843108
S=50	118.699318	124.359054	129.299781	133.996082	138.773110
S=55	123.090318	127.562168	131.434202	135.090008	138.765317
S=60	125.576395	129.128145	132.243072	135.192545	138.137225
S=65	126.984419	130.065937	132.851239	135.511516	138.145444

Table:4.2 Total expected cost rate as a function when s and S increases

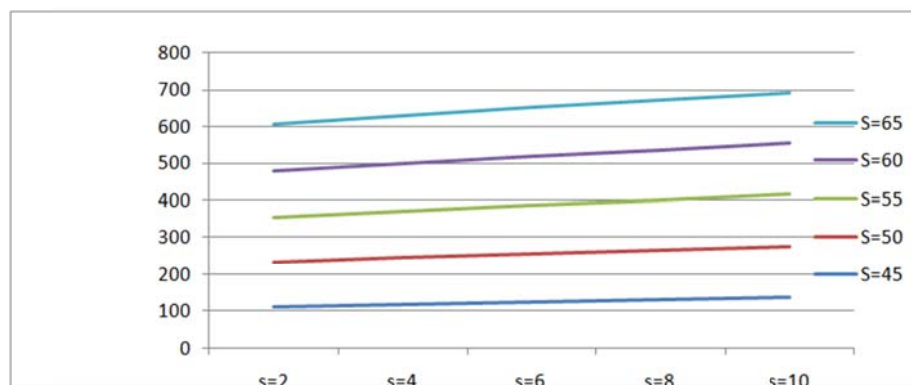


Fig.4.2: $TC(s,Q)$ for different s and S values

From the graph it is identified that the total cost increases when the s and S increases.

4.8 Conclusion

In this chapter we analyzed a continuous review inventory control system in a supply chain. The structure of the chain allows vertical movement of goods from distribution center to retailers. The model is deals with lost sales at DC and the supply from manufacturer is in terms of pockets and system handling a single perishable product. This model deals an integrated distribution system (IDS) consist with one DC and multiple retailers (n). In this model the demand at retailers node follows a independent Poisson distribution with rate λ_i ($i = 1, 2, 3, \dots, n$). Scanners collect sales data at retailer nodes and Electronic Data Interchange (EDI) allows these data to be shared to DC. With the strong communication network and transport facility a unit of item is transferred to the corresponding retailer with negligible lead time. Future research may include positive lead time for the above IDS and also we study the same system with partial backlogging.

CHAPTER 5

ANALYSIS OF INVENTORY SYSTEM WITH RETRIAL AND DIRECT DEMANDS IN SUPPLY CHAIN

5.1 Introduction

In most of the real life situations the customer may purchase the product directly from the distributor so that the price of the product may be less when compare to the retailer. But it is not possible to purchase directly from the distributor by all the customers. Therefore it is important to study the inventory system with both direct and retail demand. Many researchers are working with this type of inventory system with deterministic demand at single location. This chapter analyses the inventory system with both retail and direct demand in supply chain.

5.2 The Model description

The inventory control system in supply chain considered in this chapter is defined as follows. A product is supplied from warehouse to retailer who adopts (s, S) policy for maintaining inventory of his own. The demand at retailer node follows a Poisson process with parameter $\lambda_r (> 0)$. The unsatisfied customers at retailer are entered into the orbit of finite capacity N . These orbiting demands retry for their demand after a random time with rate $\theta (> 0)$. The Supply to the retailer in packets of $Q = S - s$ items is administrated with exponential lead time having parameter $\mu (> 0)$. We assume that the supplier (ware house) acts as a distribution center which adopts $(0, M)$ policy where $M = nQ$, $Q = S - s$, $n \leq N$. The Direct demand at Distributor node follows a Poisson distribution with parameter $\lambda_p (> 0)$. The replenishment of items in terms of pockets (from source) at distribution center is instantaneous.

In our model the maximum inventory levels M and S are fixed and reorder level s vary such that $S - s = Q$ and $M = nQ$, $n = 1, 2, \dots, [M/Q]$ where $[x]$ denotes the integral part of the real number x . We fix the following notations for the forthcoming analysis part of our chapter.

$[R]_{ij}$: The element /sub matrix at $(i,j)^{th}$ position of R .

$\mathbf{0}$: Zero matrix.

\mathbf{I} : Identity matrix.

\mathbf{e} : A column vector of 1's of appropriate dimension.

$$\sum_{i=1}^k a_i = a_1 + a_2 + \dots + a_k$$

$$\sum_{i=0}^{nQ} i = 0 + Q + 2Q + \dots + nQ$$

5.3 Analysis

Let $I_1(t)$ denote the on hand inventory level at retailer, $I_2(t)$ denote the number of customers in the orbit and $I_3(t)$ denote the on hand inventory level at Distribution center at time t^+ . From the assumptions on the input and output process,

$$\mathbf{I}(t) = \{(\mathbf{I1}(t), \mathbf{I2}(t)), \mathbf{I3}(t) : t \geq 0 \}$$

is a Markov process with state space

$$\mathbf{E} = \{(\mathbf{i}, \mathbf{j}, \mathbf{k}) / \mathbf{i} = \mathbf{S}, \mathbf{S}-1 \dots \mathbf{3}, \mathbf{2}, \mathbf{1};$$

$$\mathbf{j} = \mathbf{0}, \mathbf{1}, \dots, \mathbf{N} : \mathbf{k} = \mathbf{Q}, \mathbf{2Q} \dots \mathbf{nQ}\}.$$

The infinitesimal generator of this process

$$A = (a(i, j, k : l, m, n))_{(i, j, k)(l, m, n) \in E}$$

can be obtained using the following arguments.

- The arrival of a demand makes a state transition in the Markov Process from (i, j, k) to $(i-1, j, k)$ with intensity of transition $\lambda_r(>0)$.
- The arrival of a retrieval demand makes a state transition in the Markov Process from (i, j, k) to $(i-1, j-1, k)$ with intensity of transition $\theta (>0)$.
- The arrival of a direct demand to DC makes a state transition in the Markov Process from (i, j, k) to $(i, j, k-Q)$ with intensity of transition $\lambda_D(>0)$.

- Replenishment of inventory at retailer node makes a transition from (i, j, k) to $(i+Q, j, k-Q)$ with rate of transition $\mu (> 0)$.

The infinitesimal generator \mathbf{R} is given by

$$R = \begin{pmatrix} A & B & 0 & 0 & 0 & \dots & 0 \\ 0 & A & B & 0 & 0 & \dots & 0 \\ 0 & 0 & A & B & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ B & 0 & 0 & 0 & 0 & \dots & A \end{pmatrix}$$

The entries of the matrix $[R]_{pq}$ can be written as

$$[R] = \begin{cases} A & \text{if } q = r; r = nQ, (n-1)Q, (n-2)Q, \dots, Q. \\ B & \text{if } q = r + Q; r = (n-1)Q, (n-2)Q, \dots, 2Q. \\ B & \text{if } q = r; r = nQ \\ 0 & \text{otherwise} \end{cases}$$

Then the sub matrices of A and B are given by

$$[R] = \begin{cases} A_1 & \text{if } q = r; r = N, N-1, N-2, \dots, 1. \\ A_2 & \text{if } q = r + Q; r = N, N-1, \dots, 1. \\ A_3 & \text{if } q = r; r = 0 \\ 0 & \text{otherwise} \end{cases}$$

And

$$[B] = \begin{cases} M & \text{if } q = r \quad r = N, N-1, N-2, \dots, 1, 0 \\ 0 & \text{otherwise} \end{cases}$$

5.3.1 Transient analysis

Let $I_1(t)$ and $I_3(t)$ denote the on hand inventory level at retailer node and warehouse (distribution node) respectively, and $I_2(t)$ denote the number of customers in the orbit of size N , at time t . From the assumptions on the input and output process, we define

$$\{I(t) : t \geq 0\} = \{(I_1(t), I_2(t)), I_3(t) : t \geq 0\}$$

is a Markov process with state space

$$E = \{(i, j, k) / i = S, S-1 \dots 3, 2, 1: \\ j = 0, 1, \dots, N : k = Q, 2Q \dots nQ\}.$$

Theorem 5.1.1

The vector process $\{I(t) : t \geq 0\}$ where $I(t) = \{(I_1(t), I_2(t)), I_3(t)\}$ for $t \geq 0$ is a continuous time Markov Chain with state space

$$E = \{(i, j, k) / i = S, S-1 \dots 3, 2, 1: \\ j = 0, 1, \dots, N : k = Q, 2Q \dots nQ\}.$$

Proof: The stochastic process $\{I(t) : t \geq 0\}$ has a discrete state space with order relation ' \leq ' that $(i, j, k) \leq (l, m, n)$ if and only if $i \leq l, j \leq m$ and $k \leq n$. To prove that $\{I(t) : t \geq 0\}$ is a Markov chain, first we do a transformation for state space E to E' such that $(i, j, k) \rightarrow i+j+k \in E'$, where $E' = \{Q, Q+1, \dots, Q+S, \dots, nQ+S, nQ+S+1, nQ+S+2, \dots, nQ+S+N\}$.

Now we may realize that $\{I(t) : t \geq 0\}$ is a stochastic process with discrete state space E' . The joint distribution of random variables $\{I(t_1), I(t_2), \dots, I(t_n)\}$ and $\{I(t_1+\tau), I(t_2+\tau), \dots, I(t_n+\tau)\}$ with $\tau > 0$ (an arbitrary real number) are equal.

In particular the conditional probability

$$\Pr\{I_n = k \mid I_{n-1} = j, I_{n-2} = i, \dots, I_0 = 1\} = \Pr\{I_n = k \mid I_{n-1} = j\}$$

due to the single step transition of states in E .

Hence $\{I(t) : t \geq 0\}$ is a continuous time Markov Chain.

Define the transition probability function

$$P_{i,j,k}(l, m, n : t) = \Pr \{(I_1(t), I_2(t), I_3(t)) = (l, m, n) \mid (I_1(0), I_2(0), I_3(0)) = (i, j, k)\}$$

The corresponding transition matrix function is given by

$$P(t) = (P_{i,j,k}(l, m, n : t))_{(i,j,k)(l,m,n) \in E}$$

which satisfies the Kolmogorov- forward equation

$$\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{R}$$

where \mathbf{R} is the infinitesimal generator.

From the above equation, together with initial condition $\mathbf{P}(0) = \mathbf{I}$, the solution can be expressed in the form

$$\mathbf{P}(t) = \mathbf{P}(0)e^{\mathbf{R}t} = e^{\mathbf{R}t}$$

where the matrix expansion in power series form is

$$e^{\mathbf{R}t} = \mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{R}^n t^n}{n!}$$

case(i) : suppose that the Eigen values of \mathbf{R} are all distinct. Then from the spectral theorem of matrices, we have

$$\mathbf{R} = \mathbf{H}\mathbf{D}\mathbf{H}^{-1}$$

where \mathbf{H} is the non-singular (formed with the right Eigen vectors of \mathbf{R}) and \mathbf{D} is the diagonal matrix having its diagonal elements the eigen values of \mathbf{R} . Now 0 is an Eigen value of \mathbf{R} and if $d_i \neq 0$, $i = 1, 2, \dots, m$ are the distinct eigen values then

$$\mathbf{D} = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 \\ 0 & d_1 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & d_{m-1} & 0 \\ 0 & \dots & \dots & \dots & d_m \end{pmatrix}$$

Then we have

$$\mathbf{D}^n = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 \\ 0 & (d_1)^n & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & (d_{m-1})^n & 0 \\ 0 & \dots & \dots & \dots & (d_m)^n \end{pmatrix}$$

and

$$R^n = HD^nH^{-1}$$

Using R^n in $P(t)$ we have the explicit solution of $P(t)$ as

$$P(t) = He^{Dt}H^{-1}$$

where

$$e^{Dt} = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & e^{d_1 t} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e^{d_{m-1} t} & 0 \\ 0 & \dots & \dots & \dots & e^{d_m t} \end{pmatrix}$$

case(ii): Suppose the Eigen values of R are all not distinct, we can find a canonical representation as $R = SZS^{-1}$. From this the transition matrix $P(t)$ can be obtained in a modified form. (Medhi, J).

5.3.2 Steady state analysis:

The structure of the infinitesimal matrix R reveals that the state space E of the Markov chain $\{I(t) : t \geq 0\}$, is finite and irreducible. Let the limiting distribution of the inventory level process be defined by

$$P_{i,j}^k = \lim_{t \rightarrow \infty} \Pr \{ (I_1(t), I_2(t), I_3(t)) = (i, j, k) \}_{(i,j,k) \in E}$$

where $P_{i,j}^k$ is the steady state probability for the system be in state (i,j,k) , (Cinlar).

Let

$$P = (P_{i,j}^{nQ}, P_{i,j}^{(n-1)Q}, P_{i,j}^{(n-2)Q} \dots P_{i,j}^Q)$$

denote the steady state probability distribution where $I = 0, 1, 2, \dots, S$: and $J = 0, 1, 2, \dots, N$ for the system under consideration. For each (i,j,k) , $P_{i,j}^k$ can be obtained by

solving the matrix equation $P.R = 0$ together with normalizing condition $\sum_{i,j,k} P_{i,j}^k = 1$

Assuming $P_{i,j}^Q = a$, we Obtain the steady state probability

$$P^{iQ} = (-1)^k a(BA)^k, i = 1, 2, \dots, n; k = n - i + 1 .$$

Where

$$a = e^1 \left[\sum_{i=0}^{n-1} (-1)^i (BA^{-1})^i \right]^{-1}$$

5.4 Operating characteristics

5.4.1 Reorder rate:

Consider the event β_i of reorders at retailer node and Distribution node .Observe that β_0 events occur whenever the inventory level at retailer node reaches s whereas the β_1 -event occurs whenever the inventory level at DC reaches 0. The mean reorder rate at retailer node is given by

$$\beta_0 = \lambda_r \sum_{q=Q}^{nQ} \sum_{j=1}^N P_{s+1,j}^q$$

The reorder rate at the DC is given by

$$\beta_1 = (\mu + \lambda_D) \sum_{i=0}^S \sum_{j=0}^N P_{1,j}^q$$

5.4.2 Mean inventory level:

Let I_0 denote the mean inventory level at retailer node and I_1 denote the mean inventory level at DC in the steady state. Thus,

$$\bar{I}_0 = \sum_{q=Q}^{nQ} \sum_{j=0}^N \left(\sum_{i=0}^s i \cdot p_{s+1,j}^q \right)$$

And

$$\bar{I}_1 = \sum_{i=0}^N \sum_{j=0}^S \left(\sum_{q=Q}^{nQ} q \cdot p_{s+1,j}^q \right)$$

5.4.3 Shortage rate:

Shortages occur only at retailer node,. The shortage rate is given by

$$\alpha_0 = \lambda_r \sum_{q=Q}^{nQ} \sum_{j=1}^N P_{0,j}^q$$

5.4.4 Expected number of customer in the Orbit:

Let γ_0 denote the number of customers in the orbit which is given by

$$\gamma_0 = \lambda_r \sum_{q=Q}^{nQ} \sum_{i=0}^S \sum_{k=1}^N k.P_{0,j}^q$$

5.5 Cost analysis

In this section we analyze the cost structure for the proposed model by considering the minimization of the steady state total expected cost per unit time. There are fixed ordering costs k_1 associated with each order initiated from warehouse to source and k_0 that of initiated from retailer to warehouse (distribution) regardless of the order size.

The holding cost h_1 per unit of item per unit time at warehouse and the holding cost h_0 per unit of item per unit time at retailer node are considered. The demands occurs during stock out period are assumed to be lost and g be the shortage cost per unit shortage at retailer node. The long run expected cost rate $C(s,Q)$ is given by

$$C(s, Q) = h_0 \bar{I}_0 + h_1 \bar{I}_1 + k_0 \beta_0 + k_1 \beta_1 + \alpha_0 g + \gamma_0 b$$

where h_i is the holding cost at node i , k_i is the setup cost for node i ($i=0,1$), g is the shortage cost for unit shortage and b is the backordering cost of a demand in the orbit per unit time. Although we have not proved analytically the convexity of the cost function $C(s,Q)$, our experience with considerable number of numerical examples indicates that $C(s,Q)$ for fixed Q to be convex in s . In some cases it turned out to be an increasing function of s . Hence we adopted the numerical search procedure to

determine the optimal values s^* and b^* , consequently we obtain optimal $n^* = \left\lceil \frac{M}{Q^*} \right\rceil$

.For large number of parameters, our calculation of $C(s, Q)$ revealed a convex structure for the same.

5.6 Numerical Illustration

In the section the problem of minimizing the long run total expected cost per unit time under the following cost structure is considered for discussion. The optimum values of the system parameters s is obtained and the sensitive analysis is also done for the system.

The results we obtained in the steady state case may be illustrated through the following numerical example,

$$S = 20, N = 3, M = 80, \lambda_R = 3, \lambda_D = 2,$$

$$\mu = 3, h_R = 1.1, h_D = 1.2, k_R = 1.5, k_d = 1.3, g_R = 2.1, g_D = 2.2, C_o = 2.3.$$

The cost for different reorder level are given by

s	2	3	4	5*	6	7	8
Q	18	17	16	15	14	13	12
$TC(s, Q)$	136.41425	124.724	106.72	103.597*	111.127	130.105	145.75

Table: 5.1 Total expected cost rate as a function s and Q

For the inventory capacity S , the optimal reorder level s^* and optimal cost $TC(s, Q)$ are indicated by the symbol *. The Convexity of the cost function is given in the graph.

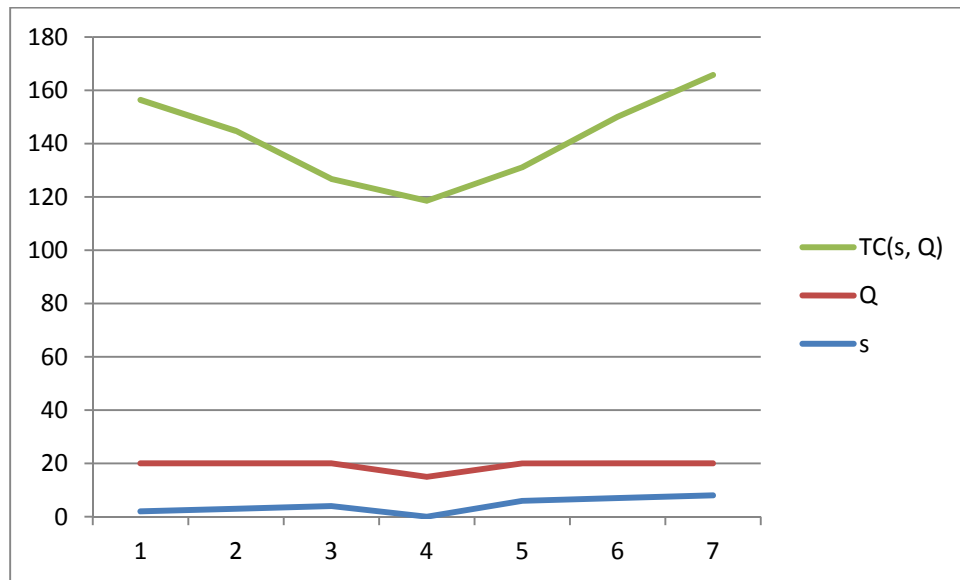


Fig 6.1: Total expected cost rate as a function $TC(s, Q)$, s and Q

5.6.1 Sensitivity Analysis

The effect of changes in Demand rate at retailer node and distributor node.

	$\lambda_D=8$	$\lambda_D=10$	$\lambda_D=12$	$\lambda_D=14$	$\lambda_D=16$
$\lambda_R=12$	45.073539	46.419603	47.214733	47.629158	47.785721
$\lambda_R=14$	46.271834	47.617898	48.413028	48.827452	48.984016
$\lambda_R=16$	47.136498	48.482562	49.277692	49.692117	49.848680
$\lambda_R=18$	47.793130	49.139194	49.934324	50.348749	50.505312
$\lambda_R=20$	48.325884	49.671948	50.467078	50.881502	51.038066

Table: 5.2 Total expected cost rate as a function when demand increases

The graph of the demand rate variation is given below and it describes, if the demand rate increases then the total cost also increases.

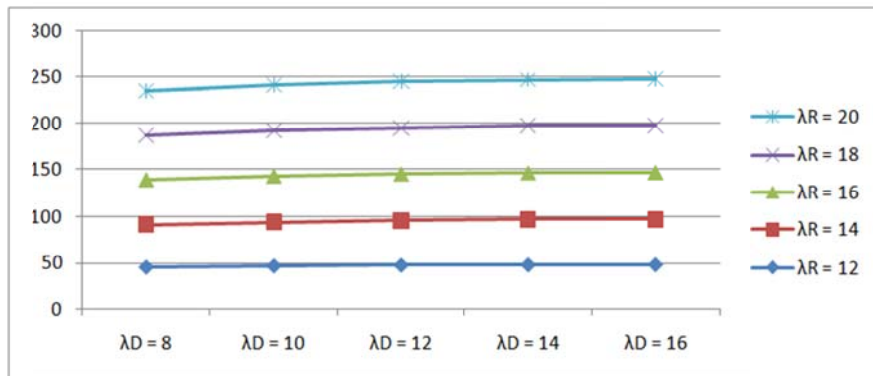


Fig.6.2: TC(s,Q) for different demand rates

Table: 5.3 Total expected cost rate as a function when s and S increases

	s=2	s=4	s=6	s=8	s=10
S=45	111.998164	118.761642	124.799813	130.664478	136.843108
S=50	118.699318	124.359054	129.299781	133.996082	138.773110
S=55	123.090318	127.562168	131.434202	135.090008	138.765317
S=60	125.576395	129.128145	132.243072	135.192545	138.137225
S=65	126.984419	130.065937	132.851239	135.511516	138.145444

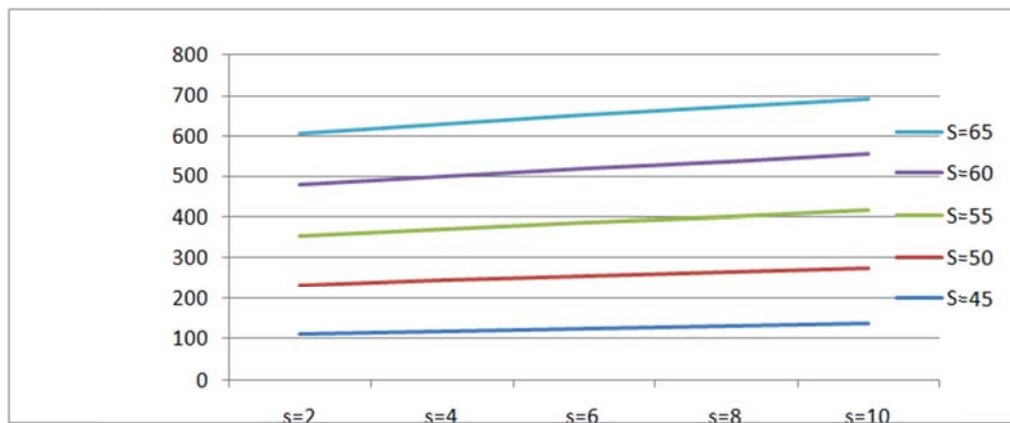


Fig.6.3: TC(s,Q) for different s and S values

From the graph it is identified that the total cost increases when the s and S increases.

	N=8	N=9	N=10	N=11	N=12
S=51	146.051244	137.918	127.412	117.153	108.63
S=52	150.008671	143.182	133.28	122.797	113.734
S=53	153.515579	148.035	139.016	128.539	118.971
S=54	156.626569	152.435	144.523	134.327	124.356
S=55	159.411166	156.374	149.71	140.088	129.882

Table: 5.4 Total expected cost rate as a function when S and N increases

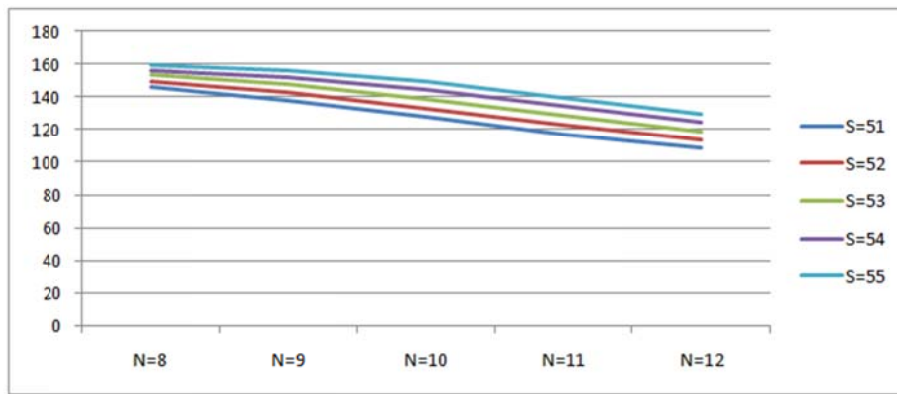


Fig.5.4: TC(s,Q) for different s and S values

From the graph it is identified that the total cost decrease when N increases and increases when S increases

	$h_D=0.04$	$h_D=0.08$	$h_D=0.12$	$h_D=0.16$	$h_D=0.20$
$h_R=0.002$	30.8517816	31.3422	31.8325	32.3229	32.8133
$h_R=0.004$	30.8598022	31.3502	31.8405	32.3309	32.8213
$h_R=0.006$	30.8678228	31.3582	31.8486	32.3389	32.8293
$h_R=0.008$	30.8758424	31.3662	31.8566	32.3469	32.8373
$h_R=0.010$	30.883863	31.3742	31.8646	32.355	32.8453

Table: 5.5. Total expected cost rate when h_R and h_D increases

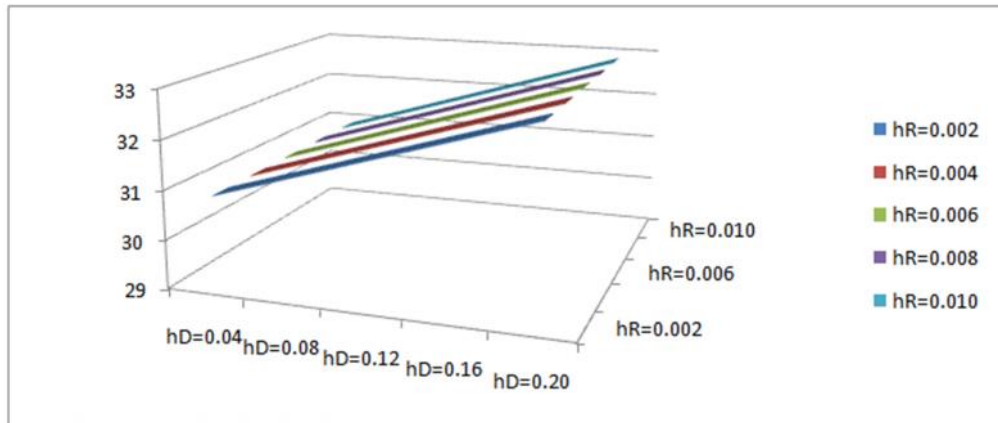


Fig.5.5: TC(s,Q) for different h_R and h_D values

As is to be expected the graph shows that the total cost increases when h_R and h_D increases

	$g_R=0.2$	$g_R=0.4$	$g_R=0.6$	$g_R=0.8$	$g_R=1.0$
$k_D=10$	23.104477	23.619900	24.135323	24.650745	25.166168
$k_D=15$	24.431902	24.947325	25.462748	25.978170	26.493593
$k_D=20$	25.759327	26.274750	26.790173	27.305595	27.821018
$k_D=25$	27.086752	27.602175	28.117598	28.633021	29.148443
$k_D=30$	28.414177	28.929600	29.445023	29.960446	30.475868

Table: 5.6. Total expected cost rate when g_R and k_D increases

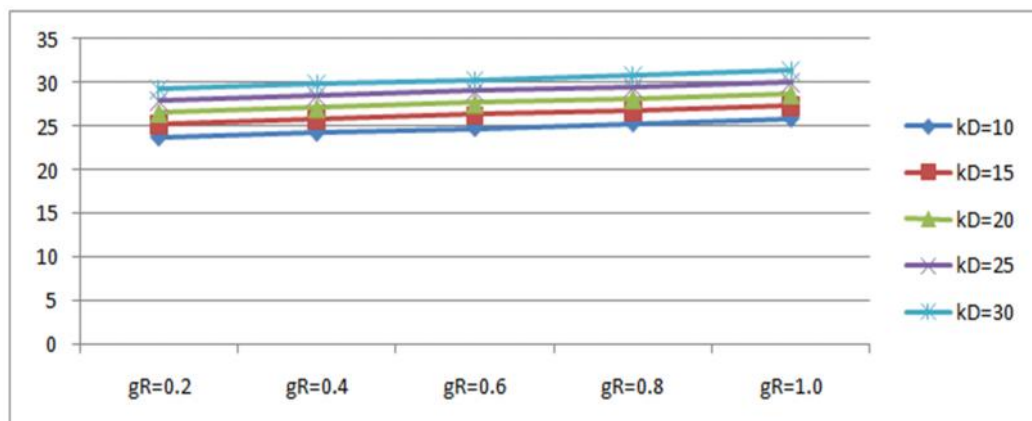


Fig.5.6: TC(s,Q) for different g_R and k_D values

As is to be expected the graph shows that the total cost increases when g_R and k_D increases.

5.7 Conclusion

In this chapter we analyzed a continuous review inventory control system with retrieval and direct demand in a supply chain. The structure of the chain allows vertical movement of goods from distribution center to retailers. The model is dealing lost sales at retailer node and the supply from distribution center is in terms of pockets. This model deals with only tandem network (basic structure of supply chain), this structure can be extended to tree structure and to more general.

CHAPTER 6

ANALYSIS OF RETRIAL DEMAND INVENTORY SYSTEM WITH PARTIAL BACKLOGGING IN SUPPLY CHAIN

6.1 Introduction

Many researchers are working with this type of inventory system with deterministic demand at single location. This chapter analyses the perishable inventory system with both retail and direct demand in supply chain. In most of the real life situations the customer may purchase the product directly from the distributor so that the price of the product may be less when compare to the retailer. But it is not possible to purchase directly from the distributor by all the customers. Therefore it is important to study this perishable inventory system with both direct and retail demand.

6.2 Model Description

In this model a three level supply chain consisting of a single perishable product, one manufacturing facility, one Distribution centre (DC) and one retailer. we assume that demands to the Distribution Centre follow Poisson process with parameter $\lambda_D(>0)$. The direct demand gets Q units at a time. The demands initiated at retailer node follow Poisson process with parameter $\lambda_R(> 0)$. It is assumed that the product is perishes in nature with exponentially distributed life time of rate $\gamma_R \geq 0$ at retailer and $\gamma_D \geq 0$ at distributor. The replenishment of Q items from DC to retailer follows exponentially distributed with parameter $\mu(>0)$. The retailer follows (s, S) policy and the distributor follow $(0, nQ)$ policy for maintaining their inventories. The unsatisfied customers are treated as retrial customers and they are waiting in the orbit with finite capacity N .

The repeated customers from the orbit (with capacity i) are entered into the system follows exponentially distributed with rate $i\theta(> 0)$. The demands occurs during stock out period are backlogged upto a finite number b . The remaining demand are treated as lost sales. Even though we have adopted two different policies in the Supply Chain, the distributors policy is depends upon the retailers policy. The model

minimizes the total cost incurred at all the locations. The system performance measures and the total cost are computed in the steady state.

6.2.1 Notations and variables

We use the following notations and variables for the analysis of the chapter.

Notations /variables	Used for
$[C]_{ij}$	The element of sub matrix at $(i,j)^{\text{th}}$ position of C
0	Zero matrix
λ_R	Mean arrival rate at retailer node
λ_D	Mean arrival rate at distributor node.
μ	Mean replacement rate at retailer from distributor
S	Maximum inventory level at retailer
N	Orbit size
s	reorder level at retailer
M	Maximum inventory at distributor
h_R	Holding cost per item at retailer
h_D	Holding cost per item at distributor
k_R	Ordering cost per order at retailer
k_D	Ordering cost per order at distributor
C_o	Waiting cost of a customer in the orbit.
g_R	Shortage cost per unit shortage at retailer.
I_R	Average inventory levelat retailer

I_D	Average inventory level at retailer distributor
r_R	Mean reorder rate at retailer
r_D	Mean reorder rate at distributor
α_R	Mean shortage rate at retailer
$\sum_{i=Q}^{nQ} i$	$Q + 2Q + 3Q + \dots + nQ$

6.3 Analysis

Let $\{ T_O(t), T_R(t), T_D(t) : t \geq 0 \}$ respectively denote the number of demands in the orbit, the on hand inventory level in the retailer node and the number of items in the Distribution centre at time t . From the assumptions on the input and output processes, clearly,

$$I(t) = \{ (T_O(t), T_R(t), T_D(t)) : t \geq 0 \}$$

is a Markov process with state space

$$E = \{ (i, j, k) / i = 0, 1, 2, \dots, N; j = S, S-1, S-2, \dots, s, s-1, \dots, 1, 0, -1, -2, \dots, -b; k = Q, 2Q, \dots, nQ \}.$$

The infinitesimal generator of this process

$$A = (a(i, j, k : l, m, n)), (i, j, k), (l, m, n) \in E$$

can be obtained from the following arguments.

- The primary arrival of demand or perish of an item to the retailer node makes a transition in the Markov process from (i, j, k) to $(i, j-1, k)$ with intensity of transition $\lambda_R + j\gamma_R$.
- The arrival of a demand at retailer node from orbit makes a transition in the Markov Process from (i, j, k) to $(i-1, j-1, k)$ with intensity of transition $i\theta$.
- The replenishment of an inventory or perish of a packet held at distributor node makes a transition from (i, j, k) to $(i, j+Q, k-Q)$ or (i, j, Q) to $(i, j+Q, nQ)$ with rate of transition μ

- The arrival of demand to the distribution centre makes a transition in the Markov process from (i, j, k) to $(i, j, k-Q)$ or (i, j, Q) to $(i, j+Q, nQ)$ with intensity of transition $\lambda_D + k\gamma_D$.

Then, the infinitesimal generator has the following finite QBD structure:

$$A = \begin{matrix} & 0 & 1 & 2 & \dots & N-2 & N-1 & N \\ \begin{matrix} 0 \\ 1 \\ 2 \\ \vdots \\ N-2 \\ N-1 \\ N \end{matrix} & \left(\begin{array}{cccccccc} A_0 & C & & & & & & \\ B_1 & A_1 & C & & & & & \\ & B_2 & A_2 & & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & & & A_{N-2} & C & \\ & & & & & B_{N-1} & A_{N-1} & C \\ & & & & & & B_N & A_N \end{array} \right) \end{matrix}$$

Where,

$$[C]_{i \times j} = \begin{cases} (\lambda_R + j\gamma_R) I_n, & j=i \quad i=0, \\ 0 & \text{otherwise} \end{cases}$$

$$[B_k]_{i \times j} = \begin{cases} i\theta I_n, & j=i-1 \quad i=1,2,\dots,S, \\ 0 & \text{otherwise} \end{cases} \quad k=0,1,2,\dots,N$$

$$[A_k]_{i \times j} = \begin{cases} (\lambda_R + j\gamma_R) I_n, & j=i-1 \quad i=S, S-1, \dots, s, s-1, \dots, 10, -1, \dots, -(b-1), \\ D, & j=i+Q \quad i=s, s-1, \dots, 10, -1, \dots, -(b-1), \\ E_k, & j=i \quad i=s, s-1, \dots, 10, -1, \dots, -(b-1), \\ F_k, & j=i \quad i=s+1, s+2, \dots, S, \\ 0, & \text{otherwise} \end{cases}$$

$$D = \begin{matrix} & Q & 2Q & 3Q & \dots & nQ \\ \begin{matrix} Q \\ 2Q \\ 3Q \\ \vdots \\ nQ \end{matrix} & \left(\begin{array}{cccccc} & & & & & \mu \\ \mu & & & & & \\ & \mu & & & & \\ & & \ddots & & & \\ & & & & & \mu \end{array} \right) \end{matrix} \quad k=0,1,2,\dots,N-1$$

$$E_k = \begin{pmatrix} Q & * & & & \lambda_D + Q\gamma_D \\ 2Q & \lambda_D + 2Q\gamma_D & * & & \\ 3Q & & \lambda_D + 3Q\gamma_D & * & \\ \vdots & & & \ddots & \\ nQ & & & & \lambda_D + nQ\gamma_D & * \end{pmatrix}$$

Where $* = -(\lambda_D + i\gamma_D + \lambda_R + i\theta + \mu)$, $i = Q, 2Q, \dots, nQ$

$$F_k = \begin{pmatrix} Q & \Omega & & & \lambda_D + Q\gamma_D \\ 2Q & \lambda_D + 2Q\gamma_D & \Omega & & \\ 3Q & & \lambda_D + 3Q\gamma_D & \Omega & \\ \vdots & & & \ddots & \\ nQ & & & & \lambda_D + nQ\gamma_D & \Omega \end{pmatrix}$$

Where $\Omega = -(\lambda_D + i\gamma_D + \lambda_R + i\theta)$, $i = Q, 2Q, \dots, nQ$ $k = 0, 1, 2, \dots, N$

Here the matrices A_k , B_k and C are the square matrices of order $((S + 1)n)$ and note that the matrix D is the square matrix of order n .

6.3.1 Transient analysis

Let $\{ T_O(t), T_R(t), T_D(t) : t \geq 0 \}$ respectively denote the number of demands in the orbit, the on hand inventory level in the retailer node and the Distribution centre at time t . From the assumptions on the input and output processes, clearly,

$$I(t) = \{ (T_O(t), T_R(t), T_D(t) : t \geq 0 \}$$

is a Markov process with state space

$$E = \{(i, j, k) / i = 0, 1, 2, \dots, N; j = S, S-1, S-2, \dots, s, s-1, \dots, 1, 0, -1, -2, \dots, -b; k = Q, 2Q, \dots, nQ\}.$$

6.3.1.1 Theorem

The vector process $\{I(t) : t \geq 0\}$ where $I(t) = \{(T_O(t), T_R(t), T_D(t), t \geq 0 \}$ is a continuous time Markov Chain with state space

$$E = \{(i, j, k) / i = 0, 1, 2, \dots, N; j = S, S-1, S-2, \dots, s, s-1, \dots, 1, 0, -1, -2, \dots, -b; k = Q, 2Q, \dots, nQ\}.$$

Proof: The stochastic process $\{I(t)=T_O(t), T_R(t), T_D(t) : t \geq 0\}$ has a discrete state space with order relation ' \leq ' that $(i, j, k) \leq (l, m, n)$ if and only if $i \leq l, j \leq m$ and $k \leq n$. To prove that $\{I(t) : t \geq 0\}$ is a Markov chain, first we do a transformation for state space E to E' such that $(i, j, k) \rightarrow i+j+k \in E'$, where

$$E' = \{Q, Q+1, \dots, Q+S, \dots, nQ+S+b, nQ+S+b+1, nQ+S+b+2, \dots, nQ+S+b+N\}.$$

Now we may realize that $\{I(t) : t \geq 0\}$ is a stochastic process with discrete state space E' .

The joint distribution of random variables

$$\{T(t_1), T(t_2), \dots, T(t_n)\} \text{ and } \{T(t_1+\tau), T(t_2+\tau), \dots, T(t_n+\tau)\}$$

with $\tau > 0$ (an arbitrary real number) are equal.

In particular the conditional probability

$$\Pr\{T_n = k \mid T_{n-1} = j, T_{n-2} = i, \dots, T_0 = 1\} = \Pr\{T_n = k \mid T_{n-1} = j\}$$

due to the single step transition of states in E .

Hence $\{I(t) : t \geq 0\}$ is a continuous time Markov Chain.

Define the transition probability function

$$P_{i,j,k}(l, m, n : t) = \Pr\{(T_O(t), T_R(t), T_D(t)) = (l, m, n) \mid (T_O(0), T_R(0), T_D(0)) = (i, j, k)\}$$

The corresponding transition matrix function is given by

$$P(t) = (P_{i,j,k}(l, m, n : t))_{(i,j,k)(l,m,n) \in E}$$

which satisfies the Kolmogorov- forward equation

$$P'(t) = P(t)A$$

where A is the infinitesimal generator. From the above equation, together with initial condition $P(0) = I$, the solution can be expressed in the form

$$P(t) = P(0)e^{At} = e^{At}$$

where the matrix expansion in power series form is $e^{At} = I + \sum_{n=1}^{\infty} \frac{A^n t^n}{n!}$

case(i) : suppose that the Eigen values of A are all distinct. Then from the spectral theorem of matrices, we have

$$A = HDH^{-1}$$

where H is the non-singular (formed with the right Eigen vectors of A) and D is the diagonal matrix having its diagonal elements the eigen values of A. Now 0 is an Eigen value of A and if $d_i \neq 0$, $i = 1, 2, \dots, m$ are the distinct eigen values then

$$D = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 \\ 0 & d_1 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & d_{m-1} & 0 \\ 0 & \dots & \dots & \dots & d_m \end{pmatrix}$$

Then we have

$$D^n = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 \\ 0 & (d_1)^n & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & (d_{m-1})^n & 0 \\ 0 & \dots & \dots & \dots & (d_m)^n \end{pmatrix}$$

and

$$A^n = HD^nH^{-1}$$

Using A^n in P(t) we have the explicit solution of P(t) as

$$P(t) = He^{Dt}H^{-1}$$

Where

$$e^{Dt} = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & e^{d_1 t} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e^{d_{m-1} t} & 0 \\ 0 & \dots & \dots & \dots & e^{d_m t} \end{pmatrix}$$

case(ii): Suppose the Eigen values of A are all not distinct, we can find a canonical representation as $A = SZS^{-1}$. From this the transition matrix P(t) can be obtained in a modified form. (Medhi, J).

6.3.2 Steady State Analysis

Since the state space is finite and irreducible, hence the stationary probability vector Π for the generator A always exists and satisfies $\Pi A = 0, \Pi e = 1$.

The vector Π can be represented by

$$\Pi = (\Pi^{<0>}, \Pi^{<1>}, \Pi^{<2>}, \dots, \Pi^{<N>}) \quad \text{Where, } 0 \leq i \leq N$$

$$\Pi^{<i>} = (\pi^{<<i,S>>}, \pi^{<<i,S-1>>}, \dots, \pi^{<<i,-b>>})$$

$$\Pi^{<<i,j>>} = (\pi^{<<<i,j,Q>>}, \pi^{<<<i,j,2Q>>}, \dots, \pi^{<<<i,j,nQ>>}),$$

$$j = S, S-1, S-2, \dots, s, s-1, \dots, 1, 0, -1, -2, \dots, -b$$

$$i = 0, 1, 2, \dots, N$$

Now the structure of A shows, the model under study is a finite birth death model in the Markovian environment. Hence we use the Gaver algorithm for computing the limiting probability vector. For the sake of completeness we provide the algorithm here.

Algorithm:

1. Determine recursively the matrix D_n , $0 \leq n \leq N$ by using
2. $D_0 = A_0$ $D_n = A_n + B_n(-D_{n-1}^{-1})C$, $n = 1, 2, \dots, K$
3. Solve the system $\Pi^{<N>} D_N = 0$.
4. Compute recursively the vector $\Pi^{<n>}$, $n = N-1, \dots, 0$ using
5. $\Pi^{<n>} = \Pi^{<n+1>} B_{n+1}(-D_n^{-1}), n = n-1, \dots, 0$.
6. Re-normalize the vector Π , using $\Pi e = 1$.

6.3.3 Performance Measures

In this section the following system performance measures in steady state for the proposed inventory system is computed.

(a) Mean Inventory level

Let I_R denote the expected inventory level in the steady state at retailer node and I_D denote the expected inventory level at distribution centre.

$$I_R = \sum_{i=0}^N \sum_{j=1}^S \sum_{k=Q}^{nQ} j \Pi^{<<<i,j,k>>>}$$

$$I_D = \sum_{i=0}^N \sum_{j=0}^S \sum_{k=Q}^{nQ} k \pi^{<<<i,j,k>>>}$$

(b) Expected Number of Demands in the Orbit

Let $E(o)$ denote the expected number of customer in the orbit which is given by

$$E(o) = \sum_{i=1}^N \sum_{j=0}^S \sum_{k=Q}^{nQ} i \Pi^{<<<i,j,k>>>}$$

(c) Mean Reorder Rate

The mean reorder rate at retailer node and distribution center is given by

$$r_R = \sum_{i=0}^N \sum_{k=Q}^{nQ} (\lambda_R + (s+1)\gamma_R + i\theta) \Pi^{<<<i,s+1,k>>>}$$

$$r_D = \mu \sum_{i=1}^N \sum_{j=0}^s \Pi^{<<<i,j,Q>>>} + (\lambda_D + \gamma_D) \sum_{i=1}^N \sum_{j=0}^S \Pi^{<<<i,j,Q>>>}$$

(d) Shortage rate

Shortage occurs only at retailer node and the shortage rate for the retailer is denoted by α_R and which is given by

$$\alpha_R = \sum_{i=0}^N \sum_{k=Q}^{nQ} \lambda_p \pi^{<<<i,-b,k>>>} + \sum_{i=1}^N \sum_{k=Q}^{nQ} i \theta \pi^{<<<i,-b,k>>>}$$

6.4 Cost Analysis

In this section the cost structure for the proposed models by considering the minimization of the steady state total expected cost per time is analyzed

The long run expected cost rate for the model is defined to be

$$TC(s, Q) = h_R I_R + h_D I_D + k_R r_R + k_D r_D + g_R \alpha_R + c_0 E(o)$$

Where,

h_R - denote the inventory holding cost/ unit / unit time at retailer node

h_D - denote the inventory holding cost/ unit / unit time at distribution centre

k_R - denote the setup cost/ order at retailer node

k_D -denote the setup cost/ order at distribution node

g_R - denote the shortage cost/ unit shortage at retailer node

c_0 - denote waiting cost of a customer in the orbit / unit time.

Although the convexity of the cost function $TC(s, Q)$, is not proved, our any experience with considerable number of numerical examples indicates that $TC(s, Q)$ for fixed $Q (> s+b+1)$ appears to be locally convex in s . For large number of parameter, the calculation of $TC(s, Q)$ revealed a convex structure.

Hence, a numerical search procedure is adopted to obtain the optimal value s for each S . Consequently, the optimal $Q (= S-s)$ and $M (= nQ)$ are obtained. A numerical example with sensitivity analysis of the optimal values by varying the different cost parameters is presented below

6.5 Numerical Illustration

In the section the problem of minimizing the long run total expected cost per unit time under the following cost structure is considered for discussion. The optimum values of the system parameters s is obtained and the sensitive analysis is also done for the system.

The results we obtained in the steady state case may be illustrated through the following numerical example

$$S = 20, N = 3, M = 80, \lambda_R = 3, \lambda_D = 2, \gamma_R = 2, \gamma_D = 3,$$

$$\mu = 3, h_R = 1.1, h_D = 1.2, k_R = 1.5, k_d = 1.3, g_R = 2.1, g_D = 2.2, C_o = 2.3$$

The cost for different reorder level are given by

s	2	3	4	5*	6	7	8
Q	18	17	16	15	14	13	12
$TC(s, Q)$	82.6753	75.5903	64.679	62.7863*	67.3499	78.8514	88.3332

Table:6.1 Total expected cost rate as a function s and Q

For the inventory capacity S , the optimal reorder level s^* and optimal cost $TC(s, Q)$ are indicated by the symbol *. The Convexity of the cost function is given in the graph.

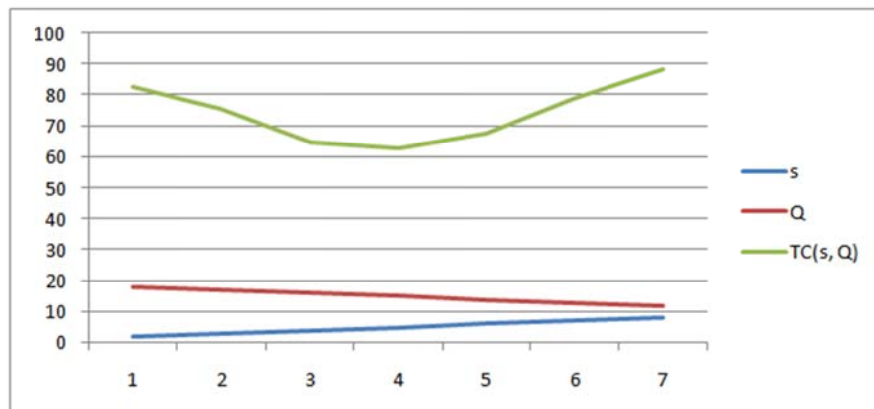


Fig 6.1:Total expected cost rate as a function $TC(s, Q)$, s and Q

6.5.1 Sensitivity Analysis

The effect of changes in Demand rate at retailer node and distributor node.

	$\lambda_D= 8$	$\lambda_D= 10$	$\lambda_D= 12$	$\lambda_D= 14$	$\lambda_D= 16$
$\lambda_R= 12$	45.073539	46.419603	47.214733	47.629158	47.785721
$\lambda_R= 14$	46.271834	47.617898	48.413028	48.827452	48.984016
$\lambda_R= 16$	47.136498	48.482562	49.277692	49.692117	49.848680
$\lambda_R= 18$	47.793130	49.139194	49.934324	50.348749	50.505312
$\lambda_R= 20$	48.325884	49.671948	50.467078	50.881502	51.038066

Table:6.2 Total expected cost rate as a function when demand increases

The graph of the demand rate variation is given below and it describes, if the demand rate increases then the total cost also increases.

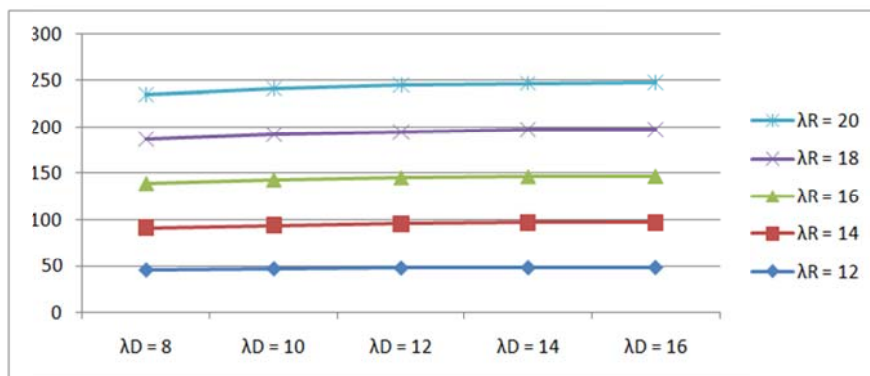


Fig.6.2: TC(s,Q) for different demand rates

	$s=2$	$s=4$	$s=6$	$s=8$	$s=10$
$S=45$	111.998164	118.761642	124.799813	130.664478	136.843108
$S=50$	118.699318	124.359054	129.299781	133.996082	138.773110
$S=55$	123.090318	127.562168	131.434202	135.090008	138.765317
$S=60$	125.576395	129.128145	132.243072	135.192545	138.137225
$S=65$	126.984419	130.065937	132.851239	135.511516	138.145444

Table:6.3 Total expected cost rate as a function when s and S increases

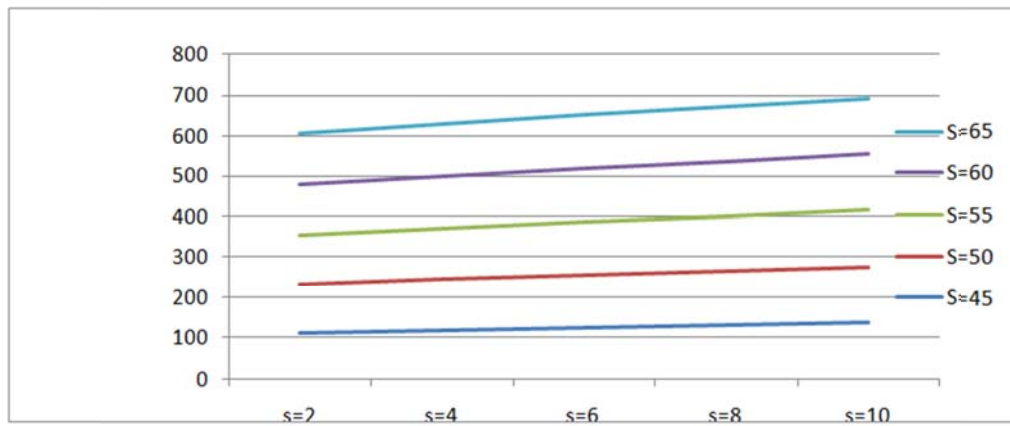


Fig.6.3: TC(s,Q) for different s and S values

From the graph it is identified that the total cost increases when the s and S increases.

	N=8	N=9	N=10	N=11	N=12
S=51	146.051244	137.918	127.412	117.153	108.63
S=52	150.008671	143.182	133.28	122.797	113.734
S=53	153.515579	148.035	139.016	128.539	118.971
S=54	156.626569	152.435	144.523	134.327	124.356
S=55	159.411166	156.374	149.71	140.088	129.882

Table:6.4 Total expected cost rate as a function when S and N increases

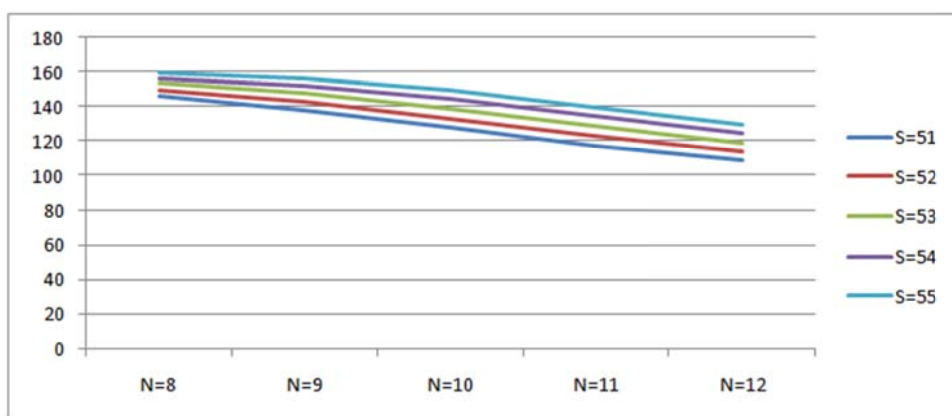


Fig.6.4: TC(s,Q) for different s and S values

From the graph it is identified that the total cost decrease when N increases and increases when S increases

	$h_D=0.04$	$h_D=0.08$	$h_D=0.12$	$h_D=0.16$	$h_D=0.20$
$h_R=0.002$	30.8517816	31.3422	31.8325	32.3229	32.8133
$h_R=0.004$	30.8598022	31.3502	31.8405	32.3309	32.8213
$h_R=0.006$	30.8678228	31.3582	31.8486	32.3389	32.8293
$h_R=0.008$	30.8758424	31.3662	31.8566	32.3469	32.8373
$h_R=0.010$	30.883863	31.3742	31.8646	32.355	32.8453

Table:6.5. Total expected cost rate when h_R and h_D increases

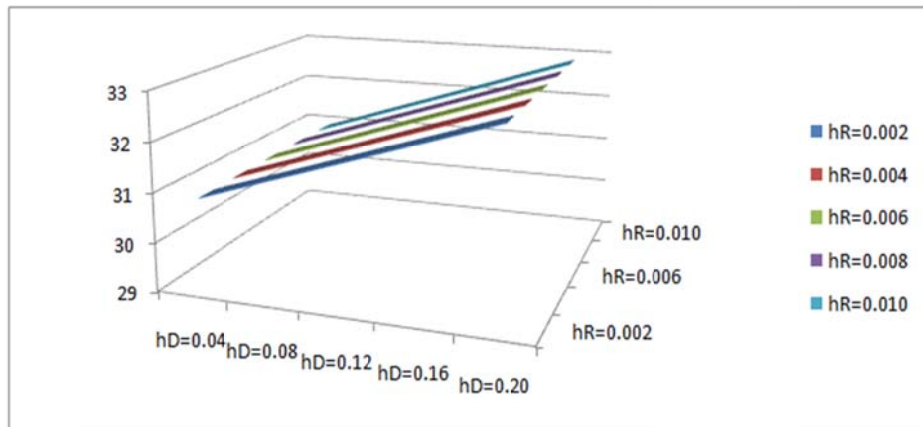


Fig.6.5: $TC(s,Q)$ for different h_R and h_D values

As is to be expected the graph shows that the total cost increases when h_R and h_D increases

	$g_R=0.2$	$g_R=0.4$	$g_R=0.6$	$g_R=0.8$	$g_R=1.0$
$k_D=10$	23.104477	23.619900	24.135323	24.650745	25.166168
$k_D=15$	24.431902	24.947325	25.462748	25.978170	26.493593
$k_D=20$	25.759327	26.274750	26.790173	27.305595	27.821018
$k_D=25$	27.086752	27.602175	28.117598	28.633021	29.148443
$k_D=30$	28.414177	28.929600	29.445023	29.960446	30.475868

Table:6.6. Total expected cost rate when g_R and k_D increases

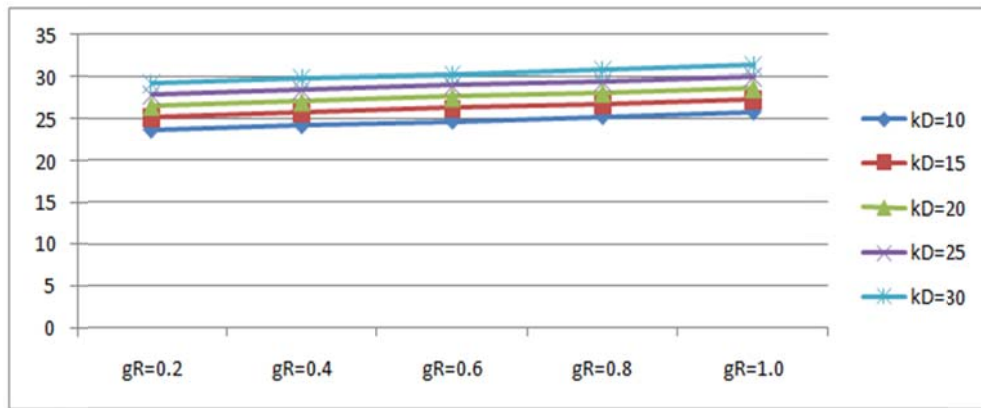


Fig.6.6: TC(s,Q) for different g_R and k_D values

As is to be expected the graph shows that the total cost increases when g_R and k_D increases.

	$\gamma_D = 0.35$	$\gamma_D = 0.40$	$\gamma_D = 0.45$	$\gamma_D = 0.50$	$\gamma_D = 0.55$
$\gamma_R = 0.2$	52.3086678	52.1322	51.9605	51.7934	51.6307
$\gamma_R = 0.3$	51.6861894	51.5098	51.338	51.1709	51.0083
$\gamma_R = 0.4$	51.1043572	50.9279	50.7562	50.5891	50.4264
$\gamma_R = 0.5$	50.5589604	50.3825	50.2108	50.0437	49.881
$\gamma_R = 0.6$	50.04675	49.8703	49.6986	49.5315	49.3688

Table:6.7 Total expected cost rate when γ_R and γ_D increases

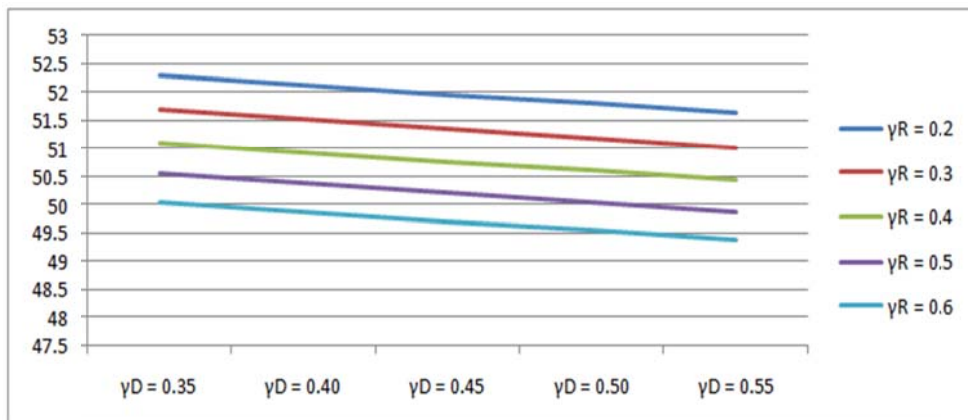


Fig.6.7: TC(s,Q) for different γ_R and γ_D values

As is to be expected the graph shows that the total cost decreases when γ_R and γ_D increases .

6.6 Conclusion.

In this model a three level supply chain consisting of a single perishable product, one manufacturing facility, one Distribution centre (DC) and one retailer. we assume that demands to the Distribution Centre follow Poisson process with parameter $\lambda_D(> 0)$. The direct demand gets Q units at a time. The demands initiated at retailer node follow Poisson process with parameter $\lambda_R(> 0)$. It is assumed that the product is perishes in nature with exponentially distributed life time of rate $\gamma_R \geq 0$ at retailer and $\gamma_D \geq 0$ at distributor. The replenishment of Q items from DC to retailer follows exponentially distributed with parameter $\mu(> 0)$. The retailer follows (s, S) policy and the distributor follow $(0, nQ)$ policy for maintaining their inventories. The unsatisfied customers are treated as retrial customers and they are waiting in the orbit with finite capacity N . The repeated customers from the orbit (with capacity i) are entered into the system follows exponentially distributed with rate $i\theta(> 0)$. The demands occurs during stock out period are backlogged up to a finite number b .

The model is analyzed within the framework of Markov processes. Joint probability distribution of inventory levels at DC, retailer and customers in the orbit in the steady state are computed. Various system performance measures are derived and the long-run expected cost rate is calculated. By assuming a suitable cost structure on the inventory system, we have presented extensive numerical illustrations to show the effect of change of values on the total expected cost rate. It would be interesting to analyze the problem discussed in this chapter by relaxing the assumption of exponentially distributed lead-times to a class of arbitrarily distributed lead-times using techniques from renewal theory and semi-regenerative processes. Once this is done, the general model can be used to generate various special cases.

CHAPTER 7

SUMMARY

7.1 Summary

This chapter depicts the summary of our study and scope for further work. In this research work several continuous review multi-echelon inventory control systems, which are building blocks for a Supply Chain Management Systems are studied

Inventory control models are almost invariably stochastic optimization problems with objectives being either minimization of risk or minimization of expected costs or maximization of expected profits. We developed new stochastic inventory models for three echelon systems and provides extensive study on the behavior of the system in the steady state.

The models proposed in this thesis include one or more of the following unique futures:

- Poisson process for demands initiated at retailer nodes is assumed.
- Retrial demand inventory system is considered for all the models.
- Exponentially distributed lead times are assumed.
- Non-perishable and Perishable products are considered in each model.
- A lost sales / partial backlogging / Retrial demand is assumed for unsatisfied demands.
- The steady state probability distribution of system states and the measures of system performance in the steady state are derived.
- Numerical illustrations are provided to compute the optimal values of s and Q .
- The sensitivity analysis for each model is examined.

Thesis Organization:

- **Chapter-1** presents, some preliminary concepts in Inventory System and its developments, importance, different type of decision-making, and Multi-echelon inventory system. Besides, a brief review of the literature relevant to the topic. An outline of the work done in the present thesis are also discussed.
- **Chapter-2**, some basic definitions and important concepts of Stochastic processes and their applications are given.
- **Chapter-3**, deals with an integrated inventory control system that consists of one manufacturer (MF), one Distribution centre (DC), and one retailer. An (s,S) type inventory system with Poisson demand and exponentially distributed lead times are assumed at retailer and DC respectively. The demands occurring during the stock out period and the unsatisfied customer demand are permitted to enter into an orbit of size N. Such unsatisfied customer demand is known as Retrial demand. The above network model is studied for both Non-perishable and Perishable products. The steady state probability distribution of system states and the measures of system performance are obtained for both the models. Numerical examples with sensitivity analysis are provided to illustrate the proposed models.
- **Chapter-4** deals a continuous review inventory control system consists of a Manufacturer (MF), single Warehouse (WH), one Distribution Centre (DC) and n identical retailers system handling a single perishable product. A (s, Q) type inventory system with Poisson demand and exponentially distributed lead times for items are assumed at DC (middle echelon). An one-for-one type inventory policy is assumed at retailers node (lower echelon). Demands occurring during the stock out periods are assumed to be lost. The WH (upper echelon) replenishes their stocks from Manufacturer, which has abundant stocks for supply. The items are supplied to the Manufacturer in packs of Q items from the warehouse. The perishable product has the exponential

decay of constant rate. Here the exposure of items occurs only at retailer node. The steady state probability distribution and the operating characteristics are obtained explicitly. The required algorithm is designed and it is executed.

- **Chapter-5**, deals with a simple supply chain that is modeled as a single warehouse and multiple retailer system handling a single product. In order to avoid the complexity, at the same time without loss of generality, we assumed identical demand pattern at each node. This restricts our study to design and analyze a tandem network of inventory, which is a building block for the whole supply chain system. A (s, S) type inventory system with Poisson demand and exponentially distributed lead times is assumed at retailer node. The items are supplied to the retailers and the direct customers (Direct demand), in packs of $Q (= S-s)$ items from the distribution center which has instantaneous replenishment facility from an abundant source (manufacturer). The unsatisfied customers at retailer are entered into the orbit of finite capacity N . These orbiting demands retry for their demand after a random time which is assumed to be exponential distribution. The transient and steady state probability distribution and the operating characteristics are obtained explicitly. The measures of system performance in the steady state are obtained. The required algorithm is designed and it is executed.
- **Chapter-6**, deals with a continuous review inventory system in supply chain (consist of retailer, distributor and manufacturer) with Markovian demand. The operating policy is (s, S) policy, that is, the maximum inventory level is S and whenever the inventory level drops to, s an order for $Q(S-s)$ units is placed. The items are supplied to the retailers and the direct customers (Direct demand), in packs of Q items from the distribution center which has instantaneous replenishment facility from an abundant source (manufacturer). The ordered items are received after a random time which is assumed to be exponential distribution. The demands that occur during stock out period are enter

into the orbit of finite size N . These orbiting demands retry for their demand after a random time, which is assumed to be exponential distribution. The joint probability distribution of the inventory levels of both retailer and distributor and the number of customers in the orbit is obtained in the steady state case. Various system performance measures in the steady state are derived and the long-run total expected cost rate is calculated. Several instances of a numerical example, which provide insight into the behavior of the system, are presented.

- **Chapter-7**, the summary of the study, conclusions and scope for further work are given.

7.2 Scope for Future Work.

The work reported in this thesis could be extended in different directions. One among them is the introduction of arbitrarily distributed demand pattern. As a first step it is possible to extend the result to renewal demand type. The model may be generalized to Markov arrivals of demands. The lead time of the system may be generalized to an arbitrary distribution. There are several interesting directions for the future research; it is proposed to formulate models with more realistic assumptions which may lead us to consider advanced techniques for the various behaviors of the Supply Chain Management System. One can develop models with multi-commodity, multiple distributors, etc.

This thesis presents an extensive numerical illustrations to show the effect of change of values for constants on the total expected cost rate. It would be interesting to analyze the problem discussed in this thesis where the life time of items are random. Naturally, with the inclusion of random life time of each items, the problem will be more challenging. Another important extension could be made by relaxing the assumption of exponentially distributed lead times to a class of arbitrarily distributed lead times using techniques from renewal theory and semi-regenerative processes. Once this is done, the general model can be used to generate various special cases. For example, three different lead time distributions one with coefficient of variation greater than one (>1), less than one (<1) and equal to one (this model) can be

compared. Cost analysis can then be carried out for (s, Q) , (s, S) and lot-for-lot models using each of the three different lead time distributions to determine which policy is optimal for any given lead time distribution.

7.3 Conclusion

In this thesis we have presented several inventory models of utility. Of these inventory with retrial of unsatisfied demands and inventory with postponed work are quite recently introduced concepts, the latter being introduced for the first time. Inventory with service time is relatively new with a handful of research work reported. The difficulty encountered in inventory with service, unlike the queueing process, is that even the simplest case needs a 2-dimensional process for its description. Only in certain specific cases we can introduce generating function to solve for the system state distribution. However numerical procedures can be developed for solving these problem. Retrial inventory, unlike retrial queues, also poses the same problem as described above. Further when an orbital customer makes a successful attempt to access the server in an inventory system with negligible service time, diagonal transitions result thereby violating the definition of a QBDP (assuming the process to be Markov). This is also the case with inventory with postponed demands (and negligible service time). In this thesis we attempted to provide performance measures of all the models discussed. However, in most cases restricted the numerical illustrations to the case in which the underlying distributions are exponential.

The work reported in this thesis could be extended in different directions. One among these is the introduction of arbitrarily distributed service time. It is also possible to have inter-arrival times of customers assumed to follow an arbitrary distribution. The inter-arrival time of orbital customers can also be assigned an arbitrary distribution provided we follow certain assumptions in the case of retrial queues. All these and more are proposed in our future investigations.

ANNEXURE

LIST OF PRESENTATIONS

1. Paper presented entitled “Analysis of Two-Echelon Inventory System with Retrieval Demand” , In the **National Conference** on Eighteenth Ramanujan Symposium On Recent Trends in Dynamical System and Mathematical Modelling, Organised by Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai.(September. 25-27, 2013).
2. Paper presented entitled “Stochastic Analysis of Perishable Inventory System in Two-Echelon”, In the CSIR Sponsored **International Conference** on Mathematical Computation and Modelling, Organised by the Madura College (Autonomous), Madurai.(January. 6, 2014).
3. Paper presented entitled “Analysis of inventory system with retrieval and direct demands in supply chain”, In the **International Conference** on Mathematical Sciences (ICMS 2014), Organised by Sathyabama University, Chennai in association with University of Central Florida, USA & The Institute of Mathematical Sciences (IMSc), Chennai.(July. 17-19, 2014). Proceedings published by **ELSEVIER**.
4. Paper presented entitled “Partial Backlogging Inventory System in Two-Echelon with Retrieval and Direct Demands” In the **National Conference** on “MATHEMATICAL MODELLING”(NCOMM-2014)organized by the Post Graduate and Research Department of Mathematics, CPA College, Bodinayakanur, Theni District, Tamilnadu on 25 & 26, September 2014.
5. Paper presented entitled “Analysis of Retrieval Demand Inventory System with Partial Backlogging in Supply Chain” In the UGC Sponsored **National Conference** on “MATHEMATICAL MODELLING”(NCOMM-2017) organized by the Post Graduate and Research Department of Mathematics, CPA College, Bodinayakanur, Theni District, Tamilnadu on March 30-31 , 2017.

LIST OF PUBLICATION

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